INTEGRAL CALCULUS

Differential calculus is about rate of change while Integral calculus is about accumulation as a result of change. For example, if s(t) is a distance covered after time t, then the speed at time t is given by $s^{/}(t)$. It should be noted that the two are related. If one is known then the other can be determined.

The process of finding a function when its derivative is known is called **integration** and the function to be found is called the **anti-derivative or the integral** of the given function. So the process of integration is the **reverse** of differentiation.

Example 1

We can say that

$x^2 + c$ is the integral of 2x with respect to x

Note that the constant c cannot be determined unless there is other data known about the behaviour of the integral. It is called the constant of integration.

Example 2

Consider $\frac{dy}{dx} = x$,

then,

$$y = \frac{1}{2}x^2 + c$$

Example 3

Consider $\frac{dy}{dx} = ax + b$

then

$$y = \frac{a}{2} x^2 + bx + c$$

Since c is arbitrary, the integral obtained is called an **Indefinite Integral.** In general, if F(x) is an anti-derivative or Integral of f(x), we say that

$$\int f(x)\, dx = F(x) + c$$

The sign $\int \dots \dots \dots .dx$ indicates the integral of the any expression …. With respect to x. In the above expression, **f(x)** is referred to as the **integrand** and **dx** is the **differential** of the variable and **F(x)** is the **integral.**

The integral is the inverse of

$$\frac{d(\ldots\ldots)}{dx}$$

which means the derivative of the bracketed expression with respect to x.

Power formula of Integration

$$\int \boldsymbol{x^n dx} = \frac{\boldsymbol{x^{n+1}}}{\boldsymbol{n+1}} + \mathbf{c}, \qquad \forall\ \mathbf{n \neq -1}$$

Note that

$$\frac{d\left(\frac{x^{n+1}}{n+1}\right)}{dx} = \frac{1}{n+1}\frac{d(\boldsymbol{x^{n+1}})}{dx} \qquad \forall\ n \neq -1$$

=> Derivative of $\frac{\boldsymbol{x^{n+1}}}{\boldsymbol{n+1}}$ is x^n.

Example 1

$\int \boldsymbol{x^4 dx} = \frac{\boldsymbol{x^{4+1}}}{\boldsymbol{4+1}} + \mathbf{c} = \frac{\boldsymbol{x^5}}{\mathbf{5}} + \mathbf{c},$

Example 2

$\int \frac{\mathbf{1}}{\boldsymbol{x^2}} \boldsymbol{dx} = \frac{\boldsymbol{x^{-2+1}}}{\mathbf{-2+1}} + \mathbf{c} = \frac{\boldsymbol{x^{-1}}}{\mathbf{-1}} + \mathbf{c} = -\frac{\mathbf{1}}{\boldsymbol{x}} + \mathbf{c}$

Example 3

$\int \boldsymbol{dx} = \frac{\boldsymbol{x^{0+1}}}{\mathbf{0+1}} + \mathbf{c} = \mathbf{x + c}$

Example 4

$\int \frac{1}{x}\,dx = \ln x + c$ (explanation $y = \ln x \Rightarrow x = e^y$. $\frac{d(x)}{dx} = \frac{d(e^y)}{dx} = e^y\frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{1}{e^y} = \frac{1}{x}$)

$$\int \frac{1}{x^2}\,dx = \frac{x^{-2+1}}{-2+1} + c = \frac{x^{-1}}{-1} + c = -\frac{1}{x} + c$$

Standard Elementary Integrals

$$\int x^n dx = \frac{x^{n+1}}{n+1} + c, \qquad \forall\ n \neq -1$$

$$\int \frac{1}{x}dx = \ln x + c \qquad \forall\ x \neq 0$$

$$\int e^x dx = e^x + c.$$

Example

$\int cf(x)dx = c \int f(x)dx$

$\int[f(x) + g(x)]dx = \int f(x)dx + \int g(x)dx$

Illustration 1

$\int \left(x - \frac{2}{x}\right)^2 dx = \int x^2 - 2x \cdot \frac{2}{x} + \frac{4}{x^2} dx = \int x^2\, dx - 4\int dx + \int \frac{4}{x^2} dx$

$= \frac{x^3}{3} - 4x - \frac{4}{x} + c$ (note $\int \frac{4}{x^2} dx = 4\frac{x^{-2+1}}{-2+1} = 4.\frac{x^{-1}}{-1} = -\frac{4}{x}$)

Illustration 2

$\int \left(\frac{2-4x+x^2+2x^3}{x^2}\right) dx = \int \frac{2}{x^2} dx - 4\int \frac{1}{x} dx + \int dx + 2\int x\, dx = -\frac{2}{x} - 4\ln|x| + x + x^2 + c$

Examples of Application

1. The marginal Revenue function is
   $R^{/}(x) = 10 - 0.02x$
   a) Find the revenue function
      Solution
      The revenue function
      $R(x) = \int(R^{/}(x))dx = \int(10 - 0.02x)dx = 10x - 0.01x^2 + c$
      When x = 0 R(x) = 0 => $10(0) - 0.01(0)^2 + c = 0$ => c = 0
      Therefore the revenue function
      $R(x) = 10x - 0.01x^2$
   b) Find the demand relation of the firm's product
      Given the product price as p, we can say that the revenue
      R(x) = price (p) . quantity supplied(x)
      => R(x) = px
      Therefore $px = 10x - 0.01x^2$
      Dividing both sides by x , we have,
      P = 10 – 0.01x, which is the demand curve,
2. The marginal profit function of a firm is $P^{/}(x) = 5 - 0.002x$ and the firm made a profit of 310 when 100 un8its were sold. What is the profit function of the firm?
      $P(x) = \int(P^{/}(x))dx = \int(5 - 0.002x)dx = 5x - 0.001x^2 + c$
   When x = 100. P =310,
   => $310 = 5(100) - 0.001(100)^2 + c$ => 310 = 500 – 10 + c => c = 310 – 490 = - 180
   Hence,
   $P(x) = 5x - 0.001x^2 - 180$

Definite Integral

A **definite integral** of a function f(x) from x=a to x= b is denoted as $\int_a^b f(x)dx$ = F(b) –F(a). From this we note

f(x) is the integrand,

F(x) is the integral,

F(a) and F(b) are the values of the integral at x= a and x= b respectively,

a and b are the lower and upper limits of the integral respectively.

We say that $\int_a^b f(x)dx$ is a definite integral because it has limits and its valuation the integral constant cancels out as shown below:

$$\int_a^b f(x)dx = [F(x)]_a^b = [F(b) + c] - [F(a) + c] = F(b) - F(a).$$

Example 1

Evaluate the following definite integral:

a) $\int_1^3 x^3 dx$

$\int_1^3 x^3 dx = [\frac{1}{4} x^4]_1^3 = \frac{1}{4}[3^4 - 1^4] = 81 - 1 = 80$

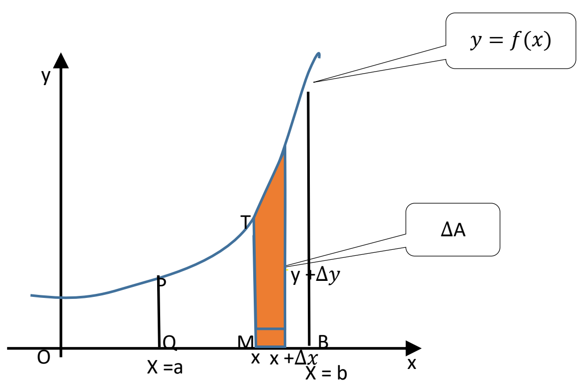
b) $\int_2^5 2x + 4\,dx = [2\frac{x^2}{2} + 4x]_2^5 = [5^2 + 4(5)]_2^5 - [2^2 + 4(2)]_2^5$

$= 4 + 8 = 12$

c) $\int_o^2 e^{2x} dx = [\frac{1}{2} e^{2x}]_1^3 = \frac{1}{2}[e^{2(3)} - e^{2(0)}] = \frac{1}{2}[e^6 - 1] = \frac{1}{2}[403.43 - 1] = 201.215.$

Evaluation Area under a curve

Consider the graph



The shaded area ΔA can fit in the following inequalities:

$$y.\,\Delta x < \Delta A(x) < (y + \Delta y).\,\Delta x$$

Dividing all through by Δx

$$y < \frac{\Delta A}{\Delta x} < (y + \Delta y)\ (y + \Delta y) \rightarrow y \text{ as } \Delta x \rightarrow 0$$

When we consider the limits of the inequalities as $\Delta x \rightarrow 0$, we have

$\lim_{\Delta x \to 0} \frac{\Delta A}{\Delta x}$ = y or A$^{/}$ (x) = y => $\frac{dA}{dx}$ = y = f(x). We therefore have,

$$A = \int y\, dx \ = \int f(x)\, dx$$

Let F(x) be the anti-derivative of f(x) then

A = F(x) + c

Now if we move TM to PQ, where x = a, then area under the curve will A = 0 =>F(a) + c =0.

=> c = -F(a).

Therefore A = F(x) – F(a)

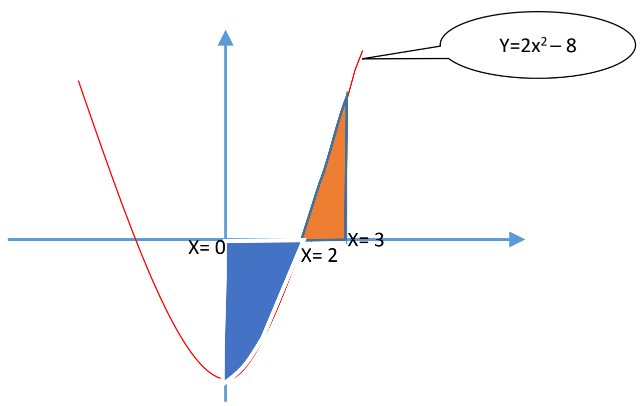
When x = b at B , the area under the curve A = F(b) – F(a) = $\int_a^b f(x)dx$ which is a definite integral. This put differently means

$$\mathbf{A = \int_a^b y dx}$$

Example

Find the area bounded by the x-axis, the curve y = f(x) = $2x^2 - 8$ and the lines x =0 and x = 3.

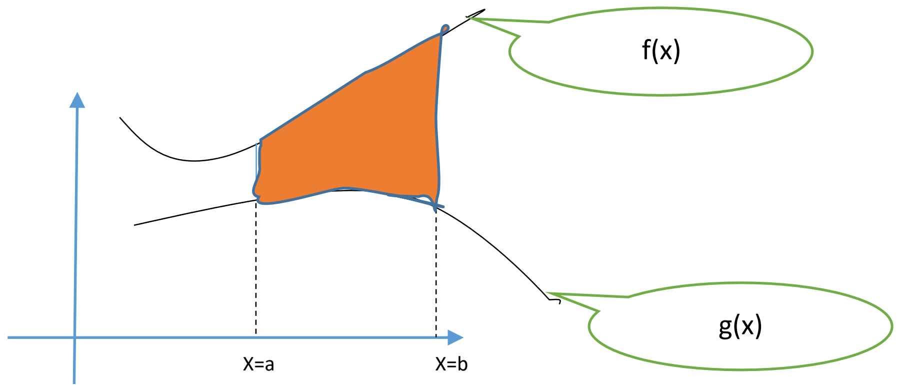
To answer this thee a need to plot the curve.



The area bounded by the x-axis, the curve y = f(x) = $2x^2 - 8$ and the lines x =0 and x = 3   is given by

$$A = -\int_0^2 2x^2 - 8\,dx + \int_2^3 2x^2 - 8\,dx = -\left[\frac{2}{3}x^3 - 8x\right]_0^2 + \left[\frac{2}{3}x^3 - 8x\right]_2^3$$

$$= -\left[\frac{16}{3} - 16\right] + \left[\left[\frac{54}{3} - 24\right] - \left(\frac{16}{3} - 16\right)\right] = 24 - \frac{32}{3} = \frac{40}{3}$$



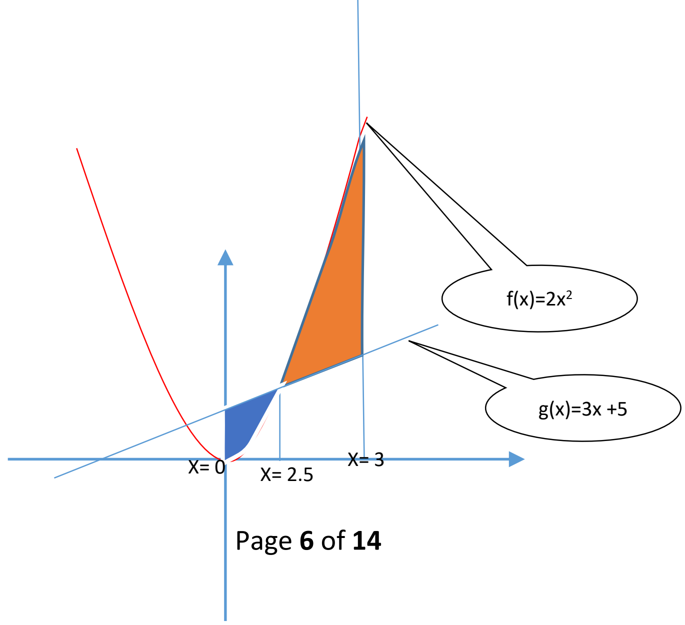
Area between two curves

The required area of the shaded part between the two curves is given by,

$A = \int_a^b f(x) - g(x)dx.$

Example,

Find the area bounded two curves y= $2x^2$ and y = 3x +5 and the lines x=0 and x=3



The area bounded by the, the curves y = f(x) = $2x^2$ and y= 3x + 5 and the lines x =0 and x = 3is given by

$A = \int_a^b f(x) - g(x)dx$. However we need to assess the trend of the relationship of the curves.

We note that the two graphs meet when

$$2x^2 = 3x + 5 \Rightarrow 2x^2 - 3x - 5 = 0$$

$$\Rightarrow 2x^2 + 2x - 5x - 5 = 0 \Rightarrow 2x(x+1) - 5(x+1) = 0$$

Hence the two graphs intersect at (-1, 2) and (2.5, 12.5). In this case the point of interest is (2.5, 12.5).

Therefore

$$A = \left[\int_0^{2.5}(3x+5) - 2x^2 dx\right] + \left[\int_{2.5}^{3} 2x^2 - (3x+5)dx\right]$$

$$= \left[\frac{3}{2}x^2 + 5x - \frac{2}{3}x^3\right]_{0}^{2.5} + \left[\frac{2}{3}x^3 - \frac{3}{2}x^2 - 5x)\right]_{2.5}^{3}$$

$$= \left[\frac{3}{2}(2.5)^2 + 5(2.5) - \frac{2}{3}(2.5)^3\right] + \left[\frac{2}{3}(3)^3 - \frac{3}{2}(3)^2 - 5(3)\right] - \left[\frac{2}{3}(2.5)^3 - \frac{3}{2}(2.5)^2 - 5(2.5)\right]$$

$$= 3(2.5)^2 - \frac{4}{3}(2.5)^3 + 10(2.5) + 18 - \frac{27}{2} - 15$$

=18.75 – 20.83 + 25 +18 – 13.5 - 15 = 12.42.

Example

Marginal cost of a certain firm is given by $C^{/}(x) = 24 - 0.03x + 0.006x^2$ if the cost of producing 200 units is UGX 22,700 find,

(a) The cost function,
(b) The fixed cost,
(c) The cost of producing 500 units

Example,

Marginal cost of a certain firm is given by $C^{/}(x) = 15.7 - 0.002x$ whereas the marginal revenue is

$R^{/}(x) = 22 - 0.004x$. Determine the increase in profit when the sales are increased from 500 to 600 units.

Example

The revenue and cost rates for an oil drilling operation are given by $R^{/}(t) = 14 - t^{1/2}$ and

$C^{/}(t) = 2 + 3t^{1/2}$ respectively, where t is time in years and R and C are in millions of dollars. How long should the drilling be continued to obtain maximum profit? What is the maximum profit?

Example

After a person has been working for t hours on a particular machine, x units will have been produced, where the rate of production (number of units per hour) is given by

$$\frac{dx}{dt} = 10(1 - e^{-t/50}).$$

How many units are produced during the person's first 50 hours on the machine? How many are produced during the second 50 hours.

Integration by substitution

This method of integration amounts to the chain rule in reverse. It is also the generalization of the power rule in the reverse. We know that

$$\int u^n\, du = \frac{u^{n+1}}{n+1} + c \qquad (n \neq -1)$$

If we put u = g(x), then du = $g'(x)dx$, and we can write that

$$\int [g(x)]^n\, g'(x)dx = \frac{(g(x))^{n+1}}{n+1} + c$$

Example 1

$$\int 2(2x+1)^5\, dx$$

If we put u = $2x + 1$ then du = 2, therefore

$\int 2(2x+1)^5\, dx = \int 2u^n\, du = 2\left(\frac{u^{5+1}}{5+1}\right) = \frac{1}{3}(2x+1)^6 + C$

Example 2

$$\int 2x\sqrt{1 + x^2}\, dx$$

$$=$$

$$\int 2x\,(1 + x^2)^{1/2}\, dx$$

Let u = 1 + x^2 => du = 2x dx , so can write

$\int 2x\,(1 + x^2)^{1/2}\, dx = \int (u)^{1/2}\, du = \frac{2}{3}(u)^{3/2} + C = \frac{2}{3}(1+x^2)^{3/2} + C$

Note:

1. Identify the existence of a function and its derivative,
2. The differential dx along with the rest of the integrand are transformed or replaced by u and du,
3. After integration the constant is added,
4. A final substitution is necessary to write in terms of variable x.

Example 3

A) Evaluate the following integral

$$\int \mathbf{4x(2x^2+1)^5\,dx}$$

Using the method of integration by substitution, we have **(1 Mark)**

Let u = $2x^2 + 1$ => du = 4x dx , so can write **(1 Mark)**

$\int 4x\,(2\,x^2+1)^5\,dx = \int (u)^5\,du = \frac{1}{6}(u)^6 + C = \frac{1}{6}(2x^2+1)^6 + C$ **(1 Mark)**

Application

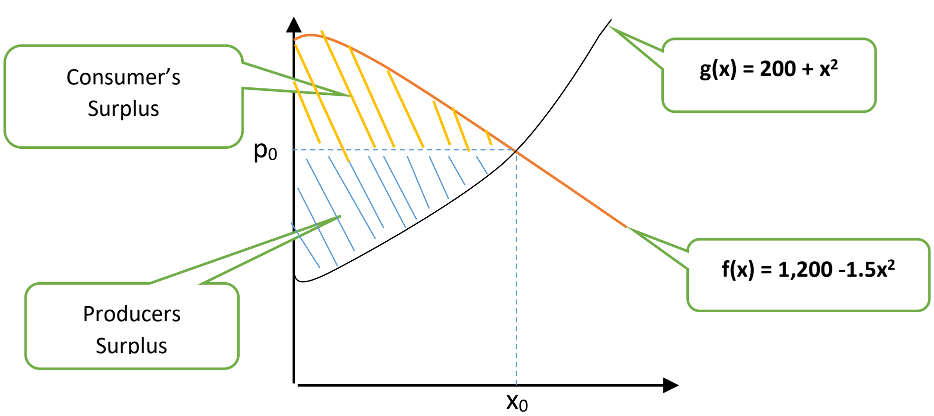
Consumers and Producers Surplus

A company has established that the supply and demand curves for its product are

Supply: **$200 + x^2$**
Demand: **$1{,}200 - 1.5x^2$**

(i) Illustrate graphically, areas showing the consumer's Surplus and Producer's surplus, hence determine their values.

Solution



(ii) In order to establish the Producer's Surplus (P.S) and Consumer's Surplus (C.S.), there is a need to establish the point (x_0, p_0) which is the market equilibrium. That is

$1{,}200 - 1.5x^2 = 200 + x^2 \Rightarrow 2.5x^2 - 1000 = 0 \Rightarrow x = 20$ or -20.
We cannot have negative number of products and therefore the feasible answer is $x_0 = 20$ => $p_0 = 200 + 400 = 600$.

C.S. = $\int_0^{x_0} \mathrm{f(x)} - \mathrm{p_0})\mathrm{dx}$ = $\int_0^{20} (\mathbf{1,200 - 1.5x^2}) - 600)\mathrm{dx}$ = $\left[600x - \frac{1.5}{3}x^3\right] \begin{matrix} 20 \\ 0 \end{matrix}$

= 12,000 – 4,000 = 8,000.

P.S. = p_0x_0 - $\int_0^{x_0} g(\mathrm{x})\mathrm{dx}$ = (20x600) - $\int_0^{20} (\mathbf{200 + x^2})\mathrm{dx}$ = 12,000 - $\left[200x + \frac{1}{3}x^3\right] \begin{matrix} 20 \\ 0 \end{matrix}$

= 12,000 –[4,000 +$\frac{8,000}{3}$] = $\frac{16,000}{3}$.

Example

The Marginal cost function for Bushenyi Traders Ltd at a level of of production x is given by

$C^/(x) = 23.5 – 0.01x$

Find the increase in total cost when the production level is increased from 1000 to 1500 units

Solution

The increase in cost is given by

$\int_{1000}^{1500} C^/(x)\ dx$

= $\int_{1000}^{1500} (23.5 - 0.01x)\ dx$ = $\left[23.5x - 0.01\left(\frac{x^2}{2}\right)\right] \begin{matrix} 1500 \\ 1000 \end{matrix}$

$= 23.5\ (1500) - 0.005(1500^{2)})$ - $23.5\ (1000) - 0.005(1000^{2)})$

= 35,250 -11,250 – (23,250 -5000) = 5,500

The cost increase is 5,500.

Example

The Marginal cost function for Bushenyi Traders Ltd at a level of production q is given by

$C^/(q) = 23.5 – 0.01q$

Find the increase in total cost when the production level is increased from 1000 to 1500 units

Solution

The increase in cost is given by

$\int_{1000}^{1500} C^/(q)\ dq$

= $\int_{1000}^{1500} (23.5 - 0.01q)\ dq$ = $\left[23.5q - 0.01\left(\frac{q^2}{2}\right)\right] \begin{matrix} 1500 \\ 1000 \end{matrix}$

$= 23.5\ (1500) - 0.005(1500^{2)})$ - $23.5\ (1000) - 0.005(1000^{2)})$

= 35,250 -11,250 – (23,250 -5000) = 5,500

The cost increase is 5,500.

Profit Evaluation

From operation records of Igara Shoe manufacturers Ltd, the marginal cost, in dollars, was found to be,

$$C^{/}(x) = x\sqrt{x^2 + 2500}.$$

where x is the numbers of pairs produced per week. If the fixed costs per week are **$ 100**, find the cost function.

Solution

The cost function C(x) = $\int C^{/}(x)dx = \int x(x^2 + 2500)^{1/2}\, dx$

Using substitution method of integration, let u = $x^2 + 2500$ => du = 2x dx

Therefore

$\int x(x^2 + 2500)^{1/2}\, dx = \int \frac{(U)^{1/2}}{2})dx = \frac{(U)^{3/2}}{3}$ + C

=> $\int x(x^2 + 2500)^{1/2}\, dx = \frac{(x^2+2500)^{3/2}}{3}$ + C

The fixed costs are when x = 0, therefore

$\frac{(0^2+2500)^{3/2}}{3}$ +C = 100 => C = 100 - $\frac{(2500)^{3/2}}{3}$ = 100 - $\frac{(50)^3}{3}$ = $\frac{300-125,000}{3}$

= $\frac{-124,700}{3}$

Hence the cost function, C(x) = $\frac{(x^2+2500)^{3/2}}{3}$ - $\frac{124,700}{3}$

Integration by parts

This is a consequence of the derivative of a product of functions.

$\frac{d(v(x)U(x))}{dx}$ = v(x)u$'$(x) + u(x)v$'$(x) => v(x)u$'$(x) = $\frac{d(v(x)U(x))}{dx}$ - u(x)v$'$(x)

$\int$ v(x)u$'$(x) dx = u(x)v(x) - $\int$ u(x)v$'$(x)dx

Or

$$\int v \frac{du}{dx} dx = uv - \int u \frac{dv}{dx} dx$$

So if we say,

v(x) = f(x) then v$'$(x) = f$'$ (x) and if u$'$(x) = g(x) then u(x) = G(x) where G(x) is the integral of g(x).

We can write,

$\int f(x)g(x)dx$ = f(x) G(x) - $\int f'(x)G(x)dx$

Example

$\int x\, e^{2x}$ dx

Let f(x) = x and g(x)= e^{2x} => f$'$(x) = 1 and G(x) = $\frac{1}{2}$ e^{2x}

Therefore,

$\int x\, e^{2x}$ dx = $\frac{x}{2}$ e^{2x} - $\int 1.\frac{1}{2}\, e^{2x}$ dx

= $\frac{x}{2}$ e^{2x} - $\frac{1}{4}$ e^{2x} + C

= $\frac{1}{4}$ (2x – 1) e^{2x} + C

Note that if you chose f(x) = e^{2x} , the new integral to the right would become difficult to integrate.

Example 2

$\int x^2\, (lnx)$ dx

Let g(x) = x^2 => G(x) = $\frac{1}{3}x^3$

f(x) = lnx => f$'$(x) = $\frac{1}{x}$

Applying

$\int f(x)g(x)dx$ = f(x) G(x) - $\int f'(x)G(x)dx$

$=> \int x^2\ (lnx)\ dx = \frac{1}{3}x^3 \ln x - \int \frac{1}{x}\frac{x^3}{3}dx\ = \frac{1}{3}x^3 \ln x - \frac{x^3}{9} + C$

Rules of integration by parts

1. If the integrand is a product of a polynomial in x and an exponential function, then it is often useful to take f(x) ie v(x) as the given polynomial,
2. If the integrand contains a logarithmic function as a factor, it is often useful to choose this as f(x) ie v(x) . If the integrand consists entirely a logarithmic function, we can take g(x) ie $\frac{du}{dx}$ as 1.

Example

$\int\ \ln(2x-1)dx$

$f(x) = \ln(2x-1) => f'(x) = \frac{2}{2x-1}$, $g(x) = 1 => G(x) = x$

$\int\ \ln(2x-1)dx = x\ln(2x-1) - \int\ \frac{2x}{2x-1}dx = x\ln(2x-1) - \int\ 1 + \frac{1}{2x-1}dx$

$= x\ln(2x-1) - x - \frac{1}{2}\ln(2x-1) + C = \frac{1}{2}(2x-1)\ln(2x-1) - x + C$

Example

The marginal revenue of a firm is $R'(x) = 100(30 - x)\ e^{-x/30}$. Find the revenue function and the demand equation of the product.

Solution

Revenue function $R(x) = \int R'(x)\ dx = \int 100(30 - x)\ e^{-x/30}\ dx$

$= 100[\int 30\ e^{-x/30}\ dx - \int\ x\,e^{-x/30}\ dx]$

$=100[\ -900e^{-x/30} - \int\ x\,e^{-x/30}\ dx]$ ………………………………………(i)

Using integration by parts consider

$\int\ x\,e^{-x/30}\ dx$

Let $f(x) = x => f'(x) = 1$

Let $g(x)\ = e^{-x/30}$ then $G(x) = -30e^{-x/30}$

Thus $\int f(x)g(x)dx\ = f(x)\,G(x) - \int f'(x)G(x)dx$

$\int\ x\,e^{-x/40}\ dx = x(-30e^{-x/30}) - \int -30\ e^{-x/30}\ dx$

$= -30x\ e^{-x/30} - 900e^{-x/30}$

Substituting in (i), we have

$R(x) = 100[\ -900e^{-x/30} + 30x\ e^{-x/30} + 900e^{-x/30}] = 3000x\ e^{-x/30} + C$

When x=0, R(0) = 0 => C=0

Thus the Revenue function is $R(x)= 3000x\ e^{-x/30}$

For the demand function,

$R(x) = xp \Rightarrow xp = 3000x\ e^{-x/30}$

$\Rightarrow p = 3000\ e^{-x/30}$

The demand function is $p = 3000\ e^{-x/30}$

Exercise

1. $\int x^{2} ln x\, dx$
2. $\int x\, (lnx)^2\, dx$
3.
4. $\int x^5\, e^{x^3}\, dx$ Hint write $x^5\, e^{x^3}$ as $(x^3(x^2)\, e^{x^3}$ then let $\frac{du}{dx} = x^2\, e^{x^3}$ use Integration by parts and then by substitution.
5. Evaluate $\int x^2\, (lnx)^2\, dx$
6. A firm has marginal cost per unit of its product given by

   $$C^{/}(x) = \frac{1000\ln(2x+40)}{(2x+40)^2}$$

   Where x is the level of production. If the fixed costs are $4000. Determine the cost function.

7. The marginal revenue of a firm is $R^{/}(x) = 10(20 - x)\, e^{-x/20}$. Find the revenue function and the demand equation of the product.