VECTOR CALCULUS

Functions with Several variables

This is a situation in which a quantity depends on more than one independent variable. This scope of functional analysis, normally falls in area referred to as **vector calculus**. Both vector-valued and real-valued functions are handled. For the scope this study, the focus will be on real-valued functions.

Definition: A real valued function of two variables also known as a bivariate function, is a rule for assigning a real number to any ordered pair (x,y) with x,y ∈ R in some set D⊆ R^2, we write that f(x,y) for the value of f with the input, (x,y).

The input x and y are called independent variables and the set D = Dom(f) is called the domain of the function f and the set of value of f over the Domain D are called the Range of f,

Range(f) = f(D) = {z∈ R| z = f(x,y) ∀ (x,y) ∈ D}

For example area of a rectangle,

A = xy where x is the length and y is the width. A = f(x,y), where x and y are the dependent variables.

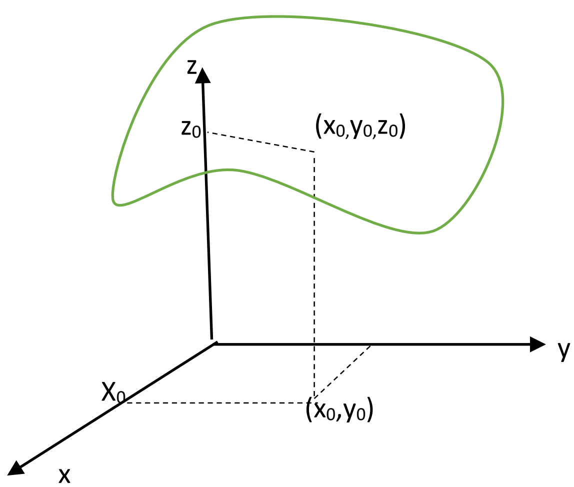
Sales is another example

Sales (R) = f (price (p), price of substitutes (s), advertising budget (A), location(d)),

R = f (p,s,A,d)

Where price (p), price of substitutes (s), advertising budget (A) and location(d) are the independent or predictor variables and R is the dependent or response variable.

Let set D contain any ordered pair of real numbers (x,y) and let **f** be the rule that specifies a unique real number for each pair, we say f is a function of two variables, x and y and the set D is the Domain of the function. The value of the function is denoted as **f**(x,y) and the set of these values is the Range of **f,** The graph of this function is a surface.



Partial Derivatives

This aspect of differential calculus concerns differentiations of functions of several independent variables.

Let a z = f(x,y) be a function with x and y as independent variables. For example is z was sales volume that may depend on price (x) and advertising budget (y). The partial derivative of z with respect to x is defined as

$$\frac{\partial z}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y) - f(x,y)}{\Delta x}.$$

This function defines the rate of change of z in the direction of x. Note that, unlike the operator for a single variable function, denoted as $\frac{d}{dy}$, for partial derivatives, the operator is $\frac{\partial}{\partial y}$.

On the other, the partial derivative of z with respect to y is defined as

$$\frac{\partial z}{\partial y} = \lim_{\Delta y \to 0} \frac{f(x, y+\Delta y) - f(x,y)}{\Delta y}$$

This means that we can calculate $\frac{\partial z}{\partial x}$ as an "ordinary" derivative by simply regarding y as a constant. In application in example above, the partial derivative of z with respect to y would mean the rate of change in sales with respect advertising budget assuming the price is constant.

Example: Calculate $\frac{\partial z}{\partial y}$ when $z = x^3 + 5x^2y - y^2 + 4xy^5$

Treating x as a constant we have,

$\frac{\partial z}{\partial y} = 0 + 5x^2 - 2y + 20xy^4 = 5x^2 - 2y + 20xy^4$.

Exercise

1. Calculate $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ for
   $z = x^4 + 5yx + 6y^2$
2. Calculate $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ for
   $z = \frac{(x^2 + 3y)}{ln\,(xy^2)}$

Notation for Partial derivatives

If z = f(x,y), then we express the partial derivatives to x and y as follows:

$$\frac{\partial z}{\partial x} = \frac{\partial f}{\partial x} = f_x(x,y) = \frac{\partial f(x,y)}{\partial x} = D_x[f(x,y)] = D_1[f(x,y)] = z_x$$

$$\frac{\partial z}{\partial y} = \frac{\partial f}{\partial y} = f_y(x,y) = \frac{\partial f(x,y)}{\partial y} = D_y[f(x,y)] = D_2[f(x,y)] = z_y$$

Example

A production function of a company has been established to take the form,

$$\mathbf{P(L,K) = cL^a K^b}$$

Where a, b and c are positive constants, and a + b = 1.

Show that $L\frac{\partial P}{\partial L} + K\frac{\partial P}{\partial K} = P(L,K)$.

Solution

Given that $P(L,K) = cL^aK^b$ where $a + b = 1$

Showing that $L\frac{\partial P}{\partial L} + K\frac{\partial P}{\partial K} = P(L,K)$

We can take L.H.S

$L.H.S = LacL^{a-1}K^b + KbcL^aK^{b-1}$

$L.H.S = acL^{1-1+a}K^b + bcK^{1-1+b}L^a$

$L.H.S = acL^aK^b + bcK^bL^a$

$L.H.S = cL^aK^b(a+b)$

$L.H.S = cL^aK^b(1)$

$L.H.S = cL^aK^b = P(L,K)$ Hence proved.

Application to Business analysis

Marginal Productivity

Example

The production function of KAMU Hides and Skins factory has been established to be;

$\mathbf{P(L,K,T)= 5L + 2L^2 T + 3 LT^2 K + 10KT + 2T^3LK^3}$

where **L** is labour in man-hours, **K** is cost of capital in millions of Shillings and **T** is thousands of power units consumed per week.

Required:

i) Determine marginal productivities when **L= 5, K = 10** and **T = 15**

ii) Explain the results in (i) above.

Solution

$$\boldsymbol{P(L,K,T) = 5L + 2L^2T + 3LT^2K + 10KT + 2T^3LK^3}$$

Cost of capital productivity

$$MPK = \frac{\partial P}{\partial K}$$

$$MPK = 3LT^2 + 10T + 6T^3LK^2$$

$$MPK_{L=5,K=10,T=15} = 3(5)(15^2) + 10(15) + 6(15^3)(5)(10)^2$$

$$MPK_{L=5,K=10,T=15} = 10{,}1128{,}525 units$$

Labour Productivity

$$MPL = \frac{\partial P}{\partial L}$$

$$MPL = 5 + 4LT + 3T^2K + 2T^3K^3$$

$$MPL_{L=5,K=10,T=15} = 5 + 4(5)(15) + 3(15^2)(10) + 2(15^3)(10^3)$$

$$MPL_{L=5,K=10,T=15} = 6{,}757{,}055\ units$$

Power Unit Productivity

$$MPT = \frac{\partial P}{\partial T}$$

$$MPT_{L=5,K=10,T=15} = 2L^2 + 6LTK + 10K + 6T^2LK^3$$

= $(2)5^2 + 6(5)(15)(10) + 10(10) + 6(15^2)(5)(10^3)$

= 6,754,650

(iii) Interpretation

For a one-unit additional increase in capital from K= 10, while Labour is held at 5 and Power units at 15, the units of output will increase by 10,128,525 units, For a one-unit additional increase in labour from L= 5, with Capital invested has been at 10 and Power at 15, the units of output will increase by 6,757,055 units and for one-unit additional increase in Power, when T = 15, with capital and labour held constant at 10 and 5 respectively, the units of output will increase by 6,754,650 units.

Example

A product is launched into a market and the sales R(x,y) increases as a function of time x (months), and also depends on the advertising budget y (thousands of shillings). The sales revenue model has been established to be

$$R(x, y) = 400(10 - e^{-0.001y})(1 - e^{-2x})$$

Calculate $\frac{\partial R}{\partial x}$ and $\frac{\partial R}{\partial y}$. Evaluate these derivatives when x = 0.5 and y = 800 and explain their practical interpretation.

Solution

We have,

$\frac{\partial R}{\partial x} = 400(10 - e^{-0.001y})(-2e^{-2x}) = 800(10 - e^{-0.001y})e^{-2x}$

$\frac{\partial R}{\partial y} = 400(0.001)e^{-0.001y}(1-e^{-2x}) = (0.4e^{-0.001y})(1-e^{-2x})$

When x = 0.5 and y = 800. Then.

$\frac{\partial R}{\partial x} = 800(10 - e^{-0.8})e^{-1} = 800(10 - 0.45)0.37 = 2{,}827$

$\frac{\partial R}{\partial y} = 0.4e^{-0.8}(10 - e^{-1}) = 0.4(0.45)(1-0.37) = 0.113$

The interpretation is that

1. $\frac{\partial R}{\partial x}$ means that when the advertising budget is fixed at 800, 000 per month then the sales volume is growing at an instantaneous rate of 2,827 units per month.
2. $\frac{\partial R}{\partial y}$ means that at the end of the first half of the month, when 800.000 has been spent on advertising, and additional dollar so spent will instantaneously increase the sales volume by 0.113 units.

Example

The production function of a firm is

$$P(L,K) = 10L + L^3K + K^2 + 5K^2L^4 + 2K$$

Where L is weekly labour units in man-hours (in hundreds) and K is investment in Millions of shillings spent per week, while P is production per week.
Determine marginal productivities per week when L = 6 and K = 15.

Labour productivities,

$\frac{\partial P}{\partial L}$ = 10 +3L^{2}K + 20K^{2}L^3 and $\frac{\partial R}{\partial K}$ = L^3 + 2K +10KL^4 + 2

When L = 6 and K = 15 then,

$\frac{\partial P}{\partial L}$ = 10 + (3)6^2(15) + 20(15^2)(6^3) = 10 + 1,620 + 972,000= 976,630.

$\frac{\partial R}{\partial K}$ = 216 + 30 +194,400 +2 = 194,648

Approximations

Approximations

In the case of a single variable function, for sufficiently small Δx, the approximation of

$f(x_0 + \Delta x) = f(x_0) + f'(x_0)\,\Delta x$.

If z = f(x,y), provided Δx and Δy are sufficiently small, then,

$f(x_0 + \Delta x . y_0 + \Delta y) \approx f(x_0, y_0) + f_x(x_0, y_0)\,\Delta x + f_y(x_0, y_0)\,\Delta y$

Example$_0$

F(x,y) = $\sqrt{x+y} + \sqrt{x-y}$

Determine the estimate of f(10.1, 5.8)

In this case we take x_0 = 10 and y_0 = 6 => Δx = 0.1 and Δy = -0.2

f(10, 6) = $\sqrt{10+6} + \sqrt{10-6}$ = 4 + 2 = 6

$f_x(10,6) = \frac{1}{2\sqrt{10+6}} + \frac{1}{2\sqrt{10-6}} = \frac{1}{8} + \frac{1}{4} = \frac{3}{8}$

$f_y(10,6) = \frac{1}{2\sqrt{10+6}} - \frac{1}{2\sqrt{10-6}} = \frac{1}{8} - \frac{1}{4} = -\frac{1}{8}$

$f(10 + 0.1, 6 - 0.2) \approx 6 + \frac{3}{8}(0.1) + \left(\frac{-1}{8}\right)(-0.2)$ = 6 + 0.0375 +0 .0250 =6 .0625

f(10.1, 5.8) ≈6 .0625

Example

If f(m,w) = (m – w)/ $\sqrt{m+w}$, find the approximate value of f(2.1,1.95) and f(4.0,5.1)

Example

By using L units of labour and K units of capital, a firm can produce P units of its product, where P= f(L,K). The firm does not know the precise form of this production function, but it does have the following information.

1. When L = 64 and K = 20, P is equal to 25,000
2. When L = 64 and K = 20, the marginal productivities of labour and capital are P_L = 270 and P_K =350.

The firm is contemplating an expansion in its plant that would change L to 69 and K to 24. Find the approximate increase in output which would result.

Example

The production function of a firm is given by

P(L,K) = $9L^{2/3}\ K^{1/3}$

Where P represents the total output when L units of labour and K units of capital are used. Approximate the total output when L = 1003 and K = 28

Second Order Partial Derivatives

The derivatives $\frac{\partial z}{\partial y}$ and $\frac{\partial z}{\partial y}$ are functions and can be differentiated. We can construct partial derivatives of partial derivatives results of which are referred to a **second –order partial** derivatives. Thus we have,

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = z_{xx} = f_{xx} \text{ for } z = f(x,y).$$

Similarly, we have,

$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = z_{yy} = f_{yy}$$

We also have the second order derivatives of the form,

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = z_{xy} = f_{xy}.$$

And

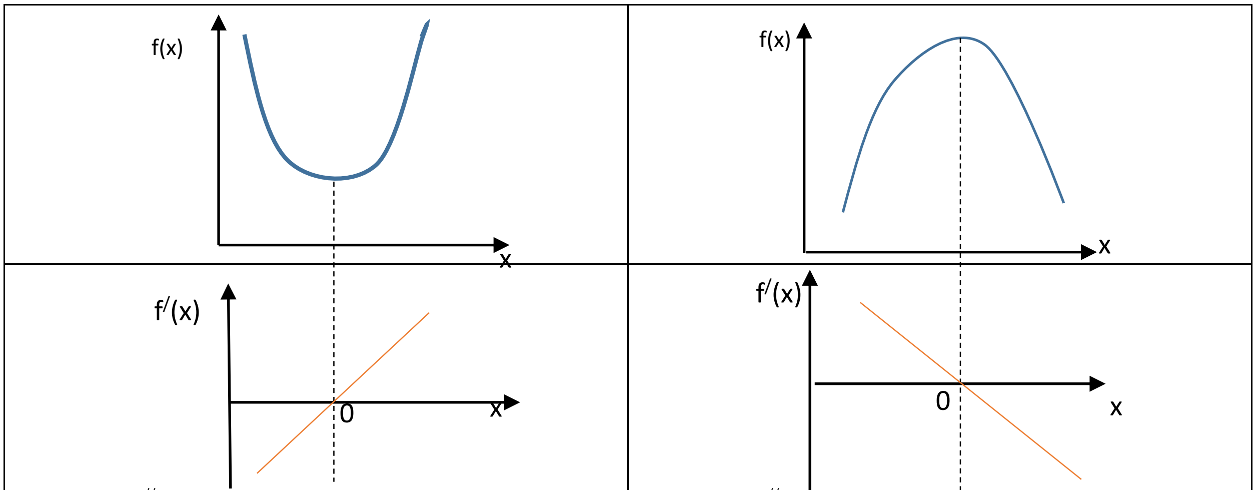
$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = z_{yx} = f_{yx}\ .$$

It is important to note that f_{xy} is the second order derivative with respect to x first and then y while f_{yx} is the second order derivative with respect to y and then x. They are referred as **Mixed Second Order partial derivatives** and they are equal if they are continuous over a given interval.

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial^2 z}{\partial x \partial y}$$

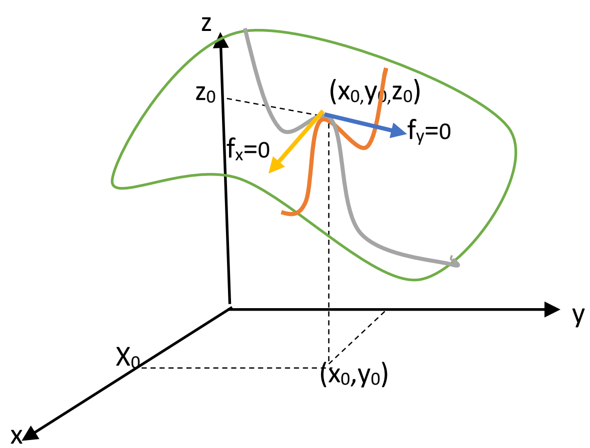
Concavity and convexity

In order to appreciate these concepts, let us look at single variable function f(x).



Because $f''(x) > 0$ we say the graph is **Convex** or **Concave upwards** – it has a minimum.	Because $f''(x) < 0$ we say the graph is **Concave** or **Convex downwards** –it has a maximum

When it comes to functions with more than one independent variable, then we are not dealing with a curve but a surface, hence the concepts of **Convexity** and **Concavity**.



As it is when a surface is **convex**, then it is likely to have a minimum. On the other hand, if it is **concave** it is likely to have a maximum.

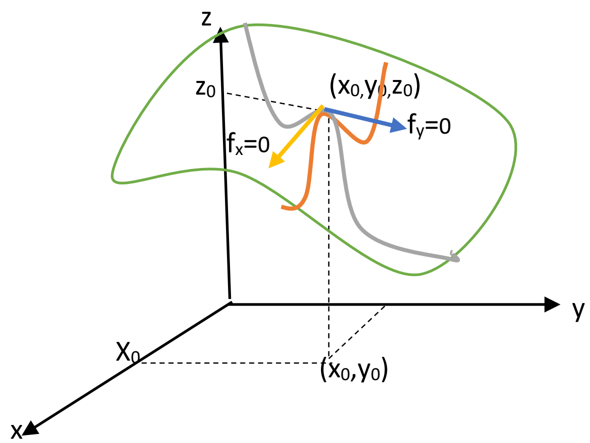
Optimization of functions of several variables

The function f(x,y) has **local minimum** point (x_0, y_0) if $f(x,y) > f(x_0, y_0)$ ∀ x and y sufficiently near x_0 and y_0. On the other hand ,the function f(x,y) has local maximum point (x_0, y_0) if f(x,y)

< $f(x_0, y_0)$ ∀ x and y sufficiently near x_0 and y_0. These two points are referred to **extremum ponts**.

If z = f(x,y), then z has a **local minimum or maximum** at point (x_0, y_0) if

$f_x(x_0, y_0) = 0$ and $f_y(x_0, y_0) = 0$



The critical point at which

$$f_x(x_0,y_0) = 0 \text{ and } f_y(x_0,y_0) = 0$$

may be a **maximum, a minimum or a saddle** point. So not every critical point is an **extremum point.**

Example

$f(x,y)= x^3 + x^2y + x - y$

$f_x = 3x^2 - 2xy + 1$……………………………………………………………………..(i)

$f_y = x^2 - 1$………………………………………………………………………………(ii)

At critical point $f_x(x,y) = f_y(x,y) = 0$.

From eqtn (ii)

$x^2 - 1 = 0 \Rightarrow x = \pm 1$

Substituting (i) , y = -2 and 2.

There are critical points

(1, -2) and (-1, 2)

There are situations when the point (x_0,y_0) is a local minimum for y-direction and a local maximum for x- direction, similar a situation of single independent variable function in case of **point of inflexion**. In a multi-variable function this is referred to as a **Saddle** point.

Thus the following conditions **differentiate** the critical points:

Define $\Delta(x,y) = f_{xx}(x,y)\, f_{yy}(x,y) - [f_{xy}(x,y)]^2$

1. For a **Local Maximum**
   $f_{xx}(x_0,y_0) < 0$ and $f_{yy}(x_0,y_0) < 0$ and $\Delta(x_0,y_0) > 0$

2. For a **Local Minimum**
   $f_{xx}(x_0,y_0) > 0$ and $f_{yy}(x_0,y_0) > 0$ and $\Delta(x_0,y_0) > 0$

3. For a **Saddle poin**t
   $\Delta(x_0,y_0) < 0$

It should be noted that when

$\Delta(x_0,y_0) = 0$,

then, this test cannot be used to establish whether the critical point is either a **minimum or maximum.**

Example

Find the local extrema of the function

$f(x,y)= x^2 + 2xy +2y^2 +2x - 2y$

At critical points

$f_x = 2x + 2y + 2 = 0$……………………………………………………………………..(i)

$f_y = 2x + 4y - 2 = 0$………………………………………………………………………(ii)

Subst in (ii)

$2y = 4 \Rightarrow y = 2$

Subst in (i)

$2x +4 +2 = 0 \Rightarrow x = -3$

The critical point is at (-3,2)

$f_{xx} = 2$, $f_{yy} = 4$ and $f_{xy} = 2$.

$\Delta(x_0,y_0) = f_{xx} f_{yy} - [f_{xy}]^2$

$= 8- 4 =4 > f_{xx} > 0$, $f_{yy} > 0$ and $\Delta > 0$

Thus the point (-3, 2) is a local minimum and $f(-3,2) = (-3)^2 +2(-3)(2) +2(-3) -2(2) = -5$

Example

The total cost per production run (in thousands of dollars) for Hides and Skins Company limited is given by

$C(x,y) = 3x^2 + 4y^2 – 5xy +3x – 14y + 20$

Where x denotes the number of man-hours and y the number (in thousands) of the pairs of shoes per run. What values of x and y will result in the minimum total cost per production run?

Solution

At the extrema,

$C_x (x,y) = C_y (x,y) =0$

$C_x (x,y) = 6x – 5y + 3 = 0$

$C_y (x,y) = 8y – 5x – 14 = 0$

Thus we have,

$6x- 5y = -3x$

$-5x + 8y = 14$

This gives a

x = 2 and y = 3

To establish that this s a minimum we test using the condition,

$f_{xx} (x_0,y_0) > 0$ and $f_{yy} (x_0,y_0) > 0$ and $\Delta (x_0,y_0) > 0$

$f_{xx} (x_0,y_0) = 6>0$, $f_{yy} (x_0,y_0) = 8 > 0$ and

$\Delta (x_0,y_0) = f_{xx} (x,y) f_{yy} (x,y) – [f_{xy}(x,y)]^2 = 6(8) – (-5)^2 =23> 0$

Therefore 2,000 man-hours and 3,000 pairs will result in minimum cost per production run

Example

Find the local extrema of the function,

$f(x,y) = x^2 + 2xy + 2y^2 +2x – 2y$

The critical points are when

$f_x(x,y) = 2x + 2y +2 = 0$

$f_y(x,y) = 2x + 4y - 2 = 0$

Bu subst.

$2y = 4 \Rightarrow y = 2$ and $x = -3$

There is a critical point at (-3,2).

To test what type extrema,

$f_{xx}(-3,2) = 2$ and $f_{yy}(-3,2) = 4$

$\Delta = f_{xx} f_{yy} - (f_{xy})^2 = 8 - 4 = 4 > 0$

This means at (-3,2)

$f_{xx} > 0$, $f_{yy} > 0$ and $\Delta > 0$

Hence it is a local minimum with a value of

$f(-3.2) = (-3)^2 + 2(-3)(2) + 2(2)^2 + 2(-3) - 2(2) = -5$

Example

It costs a company $2 per unit to manufacture its product. If A dollars per month are spent on advertising, then the number of units per month which will be sold is given by

$x = 30(1 - e^{-0.001A})(22 - p)$,

where p is the selling price. Find the value of A and p that will maximize the company's monthly profit and calculate the value of maximum profit.

Example

The total cost of C per production run (in thousands of shillings) of a certain factory is given by

$C(x,y) = 3x^2 + 4y^2 - 5xy + 3x - 14y + 20$, where x denotes the number of work-hours (in hundreds) and y the number of units (in thousands) of the product produced per run. What values of x and y will result in the minimum total cost per production run?

Example

The Peoples Confectionary Ltd makes chocolate bars in two sizes at unit cost of 10/= and 20/= respectively. The weekly demand (in thousands) is x_1 and x_2 for the two sizes are,

$x_1 = p_2 - p_1$ and $x_2 = 60 + p_1 - 3p_2$

where p_1 and p_2 the prices in shillings of the two types of chocolate bars. Determine the prices p_1 and p_2 that will maximize the company's weekly profit.

Lagrange multipliers

Named after an Italian Mathematician in 18th century. Sometimes maximization and minimization is subject to constraints. This is where Lagrange Multiplier method is used. Suppose we have an interest in extreme values of the function f(x,y,z), subject to a constraint function g(x,y,z)= 0, we can construct an **auxiliary function**

F(x,y,z, λ) = f(x,y,z) - λ g(x,y,z), where λ is a real number, and is called the Lagrange multiplier and g(x,y,z)= 0 is the constraining function. Since g(x,y,z) =0 and λ is a real number, the extremmum points of F(x,y,z, λ) are necessarily the same as those of f(x,y,z) subject to the constraint g(x,y,z)=0

At the extrema of F(x,y,z, λ), the following obtain:

$F_x = f_x - \lambda g_x = 0$

$F_y = f_y - \lambda g_y = 0$

$F_z = f_z - \lambda g_z = 0$

$F_\lambda = \quad -g_\lambda = 0$

From the above equations, it is possible to evaluate the values x,y,z and λ that correspond to the extrema of f(x,y,z), since g(x,y,z)= 0.

Example

A company can use a plant to produce two types of products, A and B. It makes a profit of 4k/= per unit of A and 6k/= per unit of B. The numbers of units of the two types that can be produced by the plant are restricted by the product transformation equation

$x^2 + y^2 + 2x + 4y - 4 = 0$

where x and y are the numbers of units in millions of A and B, produced per year. Find the amount of each type that can be produced in order to maximize the profit.

Example

By using L units of labour and K units of capital a Steel Milling limited can produce P bundles of angle bars, where

P(L, K) = $100L^{1/2} K^{2/3}$

The costs for each bundle is \$100 for the unit of labour and \$300 for the unit of capital. Steel Milling has a budgetary sum of \$ 45,000 available for production of the angle bar bundles. Using the Lagrange Multiplier method, determine the units of labour and capital that the firm should use to maximize production.

Solution

$P(L, K) = 100L^{1/2} K^{2/3}$

The **constraining function** is

100L + 300 K – 45,000 = 0

The **auxiliary** function

$P(L,K,\lambda) = 100L^{1/2} K^{2/3} - \lambda\,(100L + 300K - 45{,}000)$

To obtain maximum P(L, K) we have

$P_L = \frac{100}{2} L^{-1/2} K^{2/3} - 100\lambda = 0 \Rightarrow \lambda = \frac{1}{2} L^{-1/2} K^{2/3}$ ……………………………………………..(i)

$P_K = \frac{200}{3} L^{1/2} K^{-1/3} - 300\lambda = 0 \Rightarrow \lambda = = \frac{2}{9} L^{1/2} K^{-1/3}$……………………………………………(ii)

$P_\lambda = -(100L + 300K - 45{,}000) = 0$…………………………………………………………(iii)

From (i) and (ii), we have

$\frac{1}{2} L^{-1/2} K^{2/3} = \frac{2}{9} L^{1/2} K^{-1/3} \Rightarrow K = \frac{4}{9} L$

Subst. in (iii)

$100L + 300(\frac{4}{9} L) - 45{,}000 = 0 \Rightarrow L + \frac{4}{3} L = 450$

=> L = 193 and K = 86.