Linear Algebra Class notes

Introduction to Linear Transformations

Linear transformations are a fundamental concept in linear algebra. They are functions that take vectors as input and produce vectors as output while preserving the linear structure of the vector space. In this topic, we will give a beginner-friendly introduction to linear transformations, including their definition, properties, matrix representation, and important concepts such as linear independence, basis, rank, nullity, and change of basis.

Definition

A linear transformation is a function $T: V \rightarrow W$, where V and W are vector spaces, that satisfies two properties:

A vector space is a set whose elements, often called vectors, may be added together and multiplied by numbers called scalars. Scalars are often real numbers.

Linearity: For any vectors u and v in V, and any scalar c, $T(u + v) = T(u) + T(v)$, and $T(cu) = cT(u)$.

Homogeneity: The output vector is proportional to the input vector. In mathematical terms, if you have a vector **v** as an input and another vector **w** as an output, the relationship is expressed as:

$\mathbf{w}=\mathbf{k}\cdot\mathbf{v}$

where **k** is a constant of proportionality. This means that each component of the output vector is obtained by multiplying the corresponding component of the input vector by the same constant **k**. In this context, the vectors are said to be proportional because they maintain a consistent scaling relationship

Properties

Additivity: $T(u + v) = T(u) + T(v)$

Homogeneity: $T(cu) = cT(u)$

Terms use in Linear Transformations

Matrix Representation

Linear transformations can be represented as matrices. Given a basis for the domain V and a basis for the codomain W, the matrix representation of a linear transformation T is the matrix M whose columns are the images of the basis vectors in W.

Recall

1. ***Basis:*** *This refers to a set of linearly independent vectors that can be combined in various ways to represent any vector in the space. The basis provides a way to decompose and express vectors in a concise manner. Specifically, if you have a matrix, each column vector can be considered as a basis vector. The matrix is said to span a*

certain space if its column vectors can be combined linearly to represent any vector in that space.

2. ***Domain**: The domain of a function is the set of all possible input values for the function.*
3. ***Codomain**: The codomain is the set of all possible output values that the function can produce. It represents the full range of potential outputs.*
4. ***Range**: The range of a function is the set of actual output values that the function produces for a specific set of input values. It is a subset of the codomain.*

Here's an example to illustrate these concepts:
*Consider a function **f:R→R** defined by =**f(x)=x**^{2}. In this case:*

- ***Domain**: The domain is all real numbers R, as the function can accept any real number as an input.*
- ***Codomain**: The codomain is also R, as the function produces real numbers as outputs.*
- ***Range**: The range, however, is the set of non-negative real numbers, as the square of any real number is non-negative.*

Linear Independence and Basis

Linear independence is an important concept in linear algebra. A set of vectors is linearly independent if no vector in the set can be written as a linear combination of the others. A basis for a vector space is a linearly independent set of vectors that spans the space.

Recall

***Spanning"** refers to the idea that a set of vectors spans a vector space if every vector in that space can be expressed as a linear combination of the vectors in the set.*

Rank and Nullity

The rank of a matrix is the number of linearly independent rows or columns in the matrix. The nullity of a matrix is the dimension of the null space of the matrix, which is the set of all vectors that are mapped to the zero vector under the linear transformation represented by the matrix.

Rank Examples

Example 1

Find the rank of the 2×2 matrix

$$B = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$$

Solution:

Given,

$$B = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$$

Order of B = 2×2

$|B| = 40 - 42 = -2 \neq 0$

So, the rank of B = 2

Example 2

Given,

$$A = \begin{bmatrix} 4 & 7 \\ 8 & 14 \end{bmatrix}$$

Find the rank of matrix A.

Solution:

Given,

$$A = \begin{bmatrix} 4 & 7 \\ 8 & 14 \end{bmatrix}$$

By observing the rows, we can see that the elements of the second row are twice the elements of the first row.

$R_1 \rightarrow 2R_1 - R_2$

$$\begin{bmatrix} 0 & 0 \\ 8 & 14 \end{bmatrix}$$

Number of non-zero rows = 1

The rank of matrix A = 1.

Example 3 (echelon form by using elementary transformation)

Find the rank of the given matrix.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 5 & 7 \end{bmatrix}$$

Solution:

Given,

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 5 & 7 \end{bmatrix}$$

Now, we transform matrix A to echelon form by using elementary transformation.

$R_2 \rightarrow R_2 - 2R_1$

$R_3 \rightarrow R_3 - 3R_1$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -2 \\ 0 & -1 & -2 \end{bmatrix}$$

$R_3 \rightarrow R_3 - R_2$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

Number of non-zero rows = 2
Hence, the rank of matrix A = 2

Linear transformations

A linear transformation (or a linear map) is a function $\mathbf{T}: \mathbf{R}^n \to \mathbf{R}^m$ that satisfies the following properties:

1. $\mathbf{T}(\mathbf{x}+\mathbf{y}) = \mathbf{T}(\mathbf{x}) + \mathbf{T}(\mathbf{y})$
2. $\mathbf{T}(a\mathbf{x}) = a\mathbf{T}(\mathbf{x})$

for any vectors $\mathbf{x}, \mathbf{y} \in \mathbf{R}^n$ and any scalar $a \in \mathbf{R}$.

It is simple enough to identify whether or not a given function $\mathbf{f}(\mathbf{x})$ is a linear transformation. Just look at each term of each component of $\mathbf{f}(\mathbf{x})$. If each of these terms is a number times one of the components of $\mathbf{x}$, then $\mathbf{f}$ is a linear transformation.

Therefore, the function

$$\mathbf{f}(x, y, z) = (3x - y, 3z, 0, z - 2x)$$

is a linear transformation, while neither

$$\mathbf{g}(x, y, z) = (3x - y, 3z + 2, 0, z - 2x)$$

nor

$$\mathbf{h}(x, y, z) = (3x - y, 3xz, 0, z - 2x)$$

are linear transformations. The function $\mathbf{h}$ has a nonlinear component $3xz$ that disqualifies it. What about the function $\mathbf{g}$? It's the second component $3z + 2$ that's the problem because the term 2 is a constant that doesn't contain any components of our input vector (x, y, z).

It's easy to see that the function $\mathbf{g}$ violates the second condition above. In particular, if you set $a = 0$ in that second condition, you see that each linear transformation must satisfy

$$\mathbf{T}(\mathbf{0}) = \mathbf{0}$$

but $\mathbf{g}(0, 0, 0) = (0, 2, 0, 0)$. The condition for a linear transformation is stronger than the condition one learns in grade school for a function whose graph is a line. A single variable function $f(x) = ax + b$ is not a linear transformation unless its y-intercept b is zero.

A useful feature of a feature of a linear transformation is that there is a one-to-one correspondence between matrices and linear transformations, based on matrix vector multiplication. So, we can talk without ambiguity of **the** matrix associated with a linear transformation $\mathbf{T}(\mathbf{x})$.

Matrices and linear transformations

Let A be a 2×3 matrix, say

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 3 & 1 & 2 \end{bmatrix}.$$

What do you get if you multiply A by the vector $\mathbf{x} = (x, y, z)$? Remembering matrix multiplication, we see that

$$A\mathbf{x} = \begin{bmatrix} 1 & 0 & -1 \\ 3 & 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x - z \\ 3x + y + 2z \end{bmatrix} = (x - z, 3x + y + 2z).$$

If we define a function $\mathbf{f}(\mathbf{x}) = A\mathbf{x}$, we have created a function of three variables (x, y, z) whose output is a two-dimensional vector $(x - z, 3x + y + 2z)$. Using function notation, we can write $\mathbf{f} : \mathbf{R}^3 \to \mathbf{R}^2$. We have created a vector-valued function of three variables. So, for example, $\mathbf{f}(1, 2, 3) = (1 - 3, 3 \cdot 1 + 2 + 2 \cdot 3) = (-2, 11)$.

Given any $m \times n$ matrix B, we can define a function $\mathbf{g} : \mathbf{R}^n \to \mathbf{R}^m$ (note the order of m and n switched) by $\mathbf{g}(\mathbf{x}) = B\mathbf{x}$, where $\mathbf{x}$ is an n-dimensional vector. As another example, if

$$C = \begin{bmatrix} 5 & -3 \\ 1 & 0 \\ -7 & 4 \\ 0 & -2 \end{bmatrix},$$

then the function $\mathbf{h}(\mathbf{y}) = C\mathbf{y}$, where $y = (y_1, y_2)$, is $\mathbf{h}(\mathbf{y}) = (5y_1 - 3y_2, y_1, -7y_1 + 4y_2, -2y_2)$.

In this way, we can associate with every matrix a function. What about going the other way around? Given some function, say $\mathbf{g} : \mathbf{R}^n \to \mathbf{R}^m$, can we associate with $\mathbf{g}(\mathbf{x})$ some matrix? We can only if $\mathbf{g}(\mathbf{x})$ is a special kind of function called a linear transformation. The function $\mathbf{g}(\mathbf{x})$ is a linear transformation if each term of each component of $\mathbf{g}(\mathbf{x})$ is a number times one of the variables. So, for example, the functions $\mathbf{f}(x, y) = (2x + y, y/2)$ and $\mathbf{g}(x, y, z) = (z, 0, 1.2x)$ are linear transformation, but none of the following functions are: $\mathbf{f}(x, y) = (x^2, y, x)$, $\mathbf{g}(x, y, z) = (y, xyz)$, or $\mathbf{h}(x, y, z) = (x + 1, y, z)$. Note that both functions we obtained from matrices above were linear transformations.

Let's take the function $\mathbf{f}(x, y) = (2x + y, y, x - 3y)$, which is a linear transformation from $\mathbf{R}^2$ to $\mathbf{R}^3$. The matrix A associated with $\mathbf{f}$ will be a 3×2 matrix, which we'll write as

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}.$$

We need A to satisfy $\mathbf{f}(\mathbf{x}) = A\mathbf{x}$, where $\mathbf{x} = (x, y)$.

The easiest way to find A is the following. If we let $\mathbf{x} = (1, 0)$, then $f(\mathbf{x}) = A\mathbf{x}$ is the first column of A. (Can you see that?) So we know the first column of A is simply

$$f(1, 0) = (2, 0, 1) = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}.$$

Similarly, if $\mathbf{x} = (0, 1)$, then $f(\mathbf{x}) = A\mathbf{x}$ is the second column of A, which is

$$f(0, 1) = (1, 1, -3) = \begin{bmatrix} 1 \\ 1 \\ -3 \end{bmatrix}.$$

Putting these together, we see that the linear transformation $\mathbf{f}(\mathbf{x})$ is associated with the matrix

$$A = \begin{bmatrix} 2 & 1 \\ 0 & 1 \\ 1 & -3 \end{bmatrix}.$$

The important conclusion is that every linear transformation is associated with a matrix and vice versa.