VECTOR CALCULUS

Functions with Several variables

This is a situation in which a quantity depends on more than one independent variable. This scope of functional analysis, normally falls in area referred to as **vector calculus**. Both vector-valued and real-valued functions are handled. For the scope this study, the focus will be on real-valued functions. For example area of a rectangle,

A = xy where x is the length and y is the width. A = f(x,y), where x and y are the dependent variables.

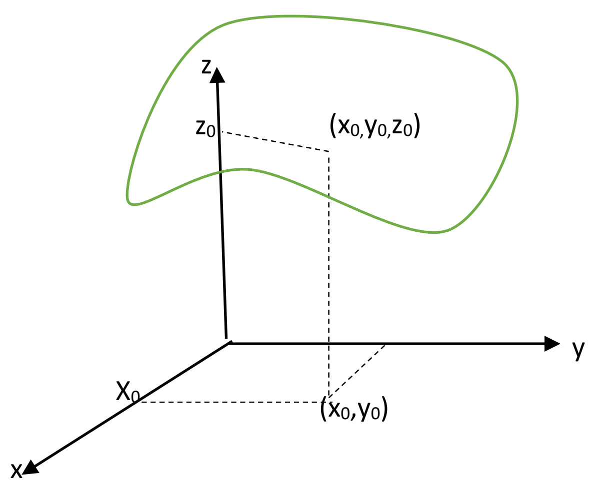
Sales is another example

Sales (R) = f (price (p), price of substitutes (s), advertising budget (A), location(d)),

R = f (p,s,A,d)

Where price (p), price of substitutes (s), advertising budget (A) and location(d) are the independent or predictor variables and R is the dependent or response variable.

Let set D contain any ordered pair of real numbers (x,y) and let **f** be the rule that specifies a unique real number for each pair, we say f is a function of two variables, x and y and the set D is the Domain of the function. The value of the function is denoted as **f**(x,y) and the set of these values is the Range of **f,** The graph of this function is a surface.



Partial Derivatives

This aspect of differential calculus concerns differentiations of functions of several independent variables.

Let a z = f(x,y) be a function with x and y as independent variables. For example is z was sales volume that may depend on price (x) and advertising budget (y). The partial derivative of z with respect to x is defined as

$$\frac{\partial z}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y) - f(x,y)}{\Delta x}.$$

This function defines the rate of change of z in the direction of x. Note that, unlike the operator for a single variable function, denoted as $\frac{d}{dy}$, for partial derivatives, the operator is $\frac{\partial}{\partial y}$.

On the other, the partial derivative of z with respect to y is defined as

$$\frac{\partial z}{\partial y} = \lim_{\Delta y \to 0} \frac{f(x, y+\Delta y) - f(x,y)}{\Delta y}$$

This means that we can calculate $\frac{\partial z}{\partial x}$ as an "ordinary" derivative by simply regarding y as a constant. In application in example above, the partial derivative of z with respect to y would mean the rate of change in sales with respect advertising budget assuming the price is constant.

Example: Calculate $\frac{\partial z}{\partial y}$ when $z = x^3 + 5x^2y - y^2 + 4xy^5$

Treating x as a constant we have,

$\frac{\partial z}{\partial y} = 0 + 5x^2 - 2y + 20xy^4 = 5x^2 - 2y + 20xy^4$.

Exercise

1. Calculate $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ for
   $z = x^4 + 5yx + 6y^2$
2. Calculate $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ for
   $z = \frac{(x^2 + 3y)}{ln\,(xy^2)}$

Notation for Partial derivatives

If z = f(x,y), then we express the partial derivatives to x and y as follows:

$$\frac{\partial z}{\partial x} = \frac{\partial f}{\partial x} = f_x(x,y) = \frac{\partial f(x,y)}{\partial x} = D_x[f(x,y)] = D_1[f(x,y)] = z_x$$

$$\frac{\partial z}{\partial y} = \frac{\partial f}{\partial y} = f_y(x,y) = \frac{\partial f(x,y)}{\partial y} = D_y[f(x,y)] = D_2[f(x,y)] = z_y$$

Example

A production function of a company has been established to take the form,

$$\mathbf{P(L,K) = cL^a K^b}$$

Where a, b and c are positive constants, and a + b = 1.

Show that $L\frac{\partial P}{\partial L} + K\frac{\partial P}{\partial K} = P(L,K)$.

ii) The production function of KAMU Hides and Skins factory has been established to be;

$$\mathbf{P(L,K,T)= 5L + 2L^2 T + 3 LT^2 K + 10KT + 2T^3LK^3}$$

where **L** is labour in man-hours, **K** is cost of capital in millions of Shillings and **T** is thousands of power units consumed per week.

Required:

i) Determine marginal productivities when **L= 5, K = 10** and **T = 15** **(3 Marks)**

ii) Explain the results in (i) above.

Solution

Given that $P(L,K) = cL^aK^b$ where $a + b = 1$

Showing that $L\frac{\partial P}{\partial L} + K\frac{\partial P}{\partial K} = P(L,K)$

We can take L.H.S

$L.H.S = LacL^{a-1}K^b + KbcL^aK^{b-1}$

$L.H.S = acL^{1-1+a}K^b + bcK^{1-1+b}L^a$

$L.H.S = acL^aK^b + bcK^bL^a$

$L.H.S = cL^aK^b(a + b)$

$L.H.S = cL^aK^b(1)$

$L.H.S = cL^aK^b = P(L,K)$ Hence proved.

(ii) $P(L,K,T) = 5L + 2L^2T + 3LT^2K + 10KT + 2T^3LK^3$

$$MPK = \frac{\partial P}{\partial K}$$

$MPK = 3LT^2 + 10T + 6T^3LK^2$

$MPK_{L=5,K=10,T=15} = 3(5)(15^2) + 10(15) + 6(15^3)(5)(10)^2$

$MPK_{L=5,K=10,T=15} = 10{,}1128{,}525 units$

$MPL = \frac{\partial P}{\partial L}$

$MPL = 5 + 4LT + 3T^2K + 2T^3K^3$

$MPL_{L=5,K=10,T=15} = 5 + 4(5)(15) + 3(15^2)(10) + 2(15^3)(10^3)$

$MPL_{L=5,K=10,T=15} = 6{,}757{,}055\ units$

$MPT = \frac{\partial P}{\partial T}$

$MPT_{L=5,K=10,T=15}$$=2L^2 + 6LTK + 10K + 6T^2LK^3$

= $(2)5^2 + 6(5)(15)(10) + 10(10) + 6(15^2\,)(5)(10^3)$

= 6,754,650

(iii) Interpretation

For a one-unit additional increase in capital from K= 10, while Labour is held at 5 and Power units at 15, the units of output will increase by 10,128,525 units, For a one-unit additional increase in labour from L= 5, with Capital invested has been at 10 and Power at 15, the units of output will increase by 6,757,055 units and for one-unit additional increase in Power, when T = 15, with capital and labour held constant at 10 and 5 respectively, the units of output will increase by 6,754,650 units.

Application to Business analysis

A product is launched into a market and the sales R(x,y) increases as a function of time x (months), and also depends on the advertising budget y (thousands of shillings). The sales revenue model has been established to be

$$R(x, y) = 400(10 - e^{-0.001y})(1- e^{-2x})$$

Calculate $\frac{\partial R}{\partial x}$ and $\frac{\partial R}{\partial y}$. Evaluate these derivatives when x = 0.5 and y = 800 and explain their practical interpretation.

Solution

We have,

$\frac{\partial R}{\partial x}$ = $400(10 - e^{-0.001y})(-2\,e^{-2x}) = 800(10 - e^{-0.001y})e^{-2x}$

$\frac{\partial R}{\partial y}$ = $400(0.001)\,e^{-0.001y}(1-e^{-2x}) = (0.4\,e^{-0.001y})(1-e^{-2x})$

When x = 0.5 and y = 800. Then.

$\frac{\partial R}{\partial x}$ = 800(10 – e $^{-0.8}$) e^{-1} = 800(10 – 0.45)0.37= 2,827

$\frac{\partial R}{\partial y}$ = 0.4e$^{-0.8}$(10 - e $^{-1}$) =0.4(0.45) (1-0.37) = 0.113

The interpretation is that

1. $\frac{\partial R}{\partial x}$ means that when the advertising budget is fixed at 800, 000 per month then the sales volume is growing at an instantaneous rate of 2,827 units per month.
2. $\frac{\partial R}{\partial y}$ means that at the end of the first half of the month, when 800.000 has been spent on advertising, and additional dollar so spent will instantaneously increase the sales volume by 0.113 units.

Marginal productivity

The production function of a firm is

$$P(L,K) = 10L + L^3K + K^2 + 5K^2L^4 + 2 K$$

Where L is weekly labour units in man-hours (in hundreds) and K is investment in Millions of shillings spent per week, while P is production per week.
Determine marginal productivities per week when L = 6 and K = 15.

Labour productivities,

$$\frac{\partial P}{\partial L} = 10 + 3L^2K + 20K^2L^3 \text{ and } \frac{\partial R}{\partial K} = L^3 + 2K + 10KL^4 + 2$$

When L = 6 and K = 15 then,

$\frac{\partial P}{\partial L}$ = 10 + (3)6^2(15) + 20(15^2)(6^3) = 10 + 1,620 + 972,000= 976,630.

$\frac{\partial R}{\partial K}$ = 216 + 30 +194,400 +2 = 194,648

Second Order Partial Derivatives

The derivatives $\frac{\partial z}{\partial y}$ and $\frac{\partial z}{\partial y}$ are functions and can be differentiated. We can construct partial derivatives of partial derivatives results of which are referred to a second –order partial derivatives. Thus we have,

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = z_{xx} = f_{xx} \text{ for } z = f(x,y).$$

Similarly, we have,

$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = z_{yy} = f_{yy}$$

We also have the second order derivatives of the form,

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = z_{xy} = f_{xy}.$$

And

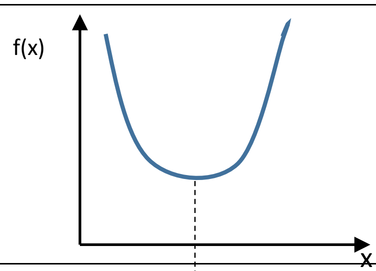
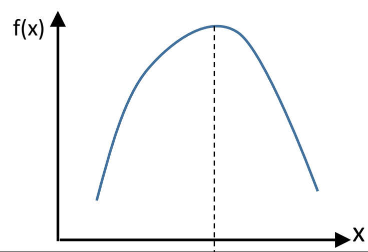
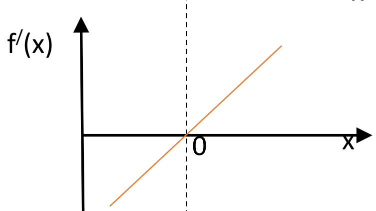
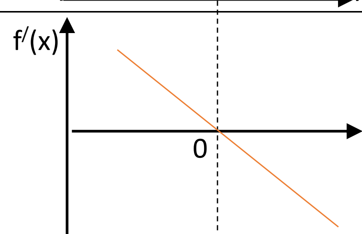
$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = z_{yx} = f_{yx}\ .$$

It is important to note that f_{xy} is the second order derivative with respect to x first and then y while f_{yx} is the second order derivative with respect to y and then x. They are referred as Mixed Second Order partial derivatives and they are equal if they are continuous over a given interval.

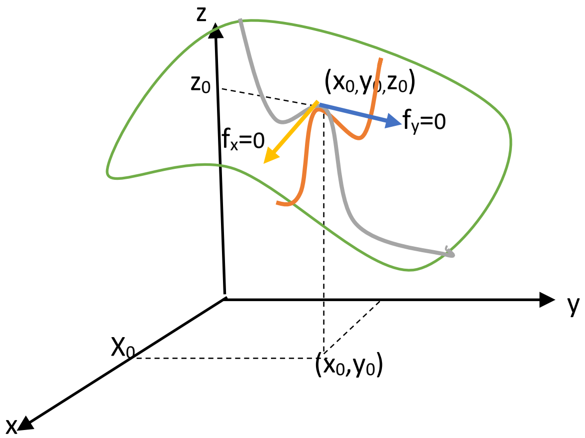
$$\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial^2 z}{\partial x \partial y}$$

Concavity and convexity

In order to appreciate these concepts, let us look at single variable function f(x).

	
 Because $f''(x) > 0$ we say the graph is Convex or Concave upwards – it has a minimum.	 Because $f''(x) < 0$ we say the graph is Concave or Concave downwards –it has a maximum

When it comes to functions with more than one independent variable, then we are not dealing with a curve but a surface, hence the concepts of Convexity and Concavity.



As it is when a surface is convex, then it likely to have a minimum. On the other hand if it is concave it is likely to have a maximum.

Optimization of functions of several variables

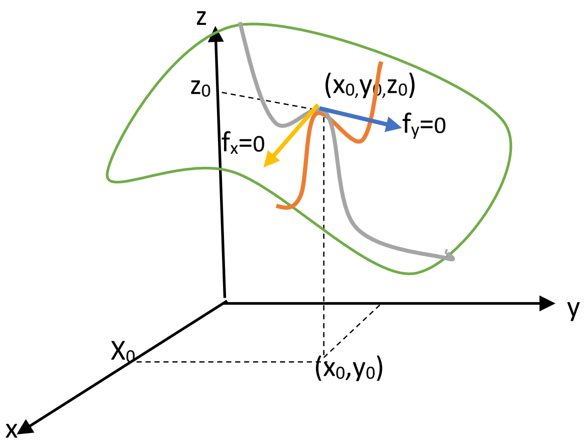
The function $f(x,y)$ has local minimum point (x_0, y_0) if $f(x,y) > f(x_0, y_0)$ $\forall$ x and y sufficiently near x_0 and y_0. On the other hand ,the function $f(x,y)$ has local miximum point (x_0, y_0) if $f(x,y) < f(x_0, y_0)$ $\forall$ x and y sufficiently near x_0 and y_0. These two points are referred to extremum ponts.

If $z = f(x,y)$, then z has a local minimum or maximum at point (x_0, y_0) if

$f_x(x_0, y_0) = 0$ and $f_y(x_0, y_0) =0$

If $z = f(x,y)$, then has a local minimum or maximum at (x_0, y_0), if

$$f_x (x_0,y_0) = 0 \text{ and } f_y (x_0,y_0) = 0$$



The critical point at which

$$f_x(x_0,y_0) = 0 \text{ and } f_y(x_0,y_0) = 0$$

may be a **maximum, a minimum or a saddle** point. So not every critical point is an extremum point.

Example

$f(x,y)= x^3 + x^2y + x - y$

$f_x = 3x^2 - 2xy + 1$……………………………………………………………………..(i)

$f_y = x^2 - 1$…………………………………………………………………………(ii)

At critical point $f_x(x,y) = f_y(x,y) = 0$.

From eqtn (ii)

$x^2 - 1 = 0 \Rightarrow x = \pm 1$

Substituting (i) , y = -2 and 2.

There are critical points

(1, -2) and (-1, 2)

There are situations when the point (x_0,y_0) is a local minimum for y-direction and a local maximum for x- direction, similar a situation of single independent variable function in case of **point of inflexion**. In a multi-variable function this is referred to as a **Saddle** point.

Thus the following conditions differentiate the critical points:

Define $\Delta(x,y) = f_{xx}(x,y)\, f_{yy}(x,y) - [f_{xy}(x,y)]^2$

1. For a Local Maximum
   $f_{xx}(x_0,y_0) < 0$ and $f_{yy}(x_0,y_0) < 0$ and $\Delta(x_0,y_0) > 0$

2. For a Local Minimum
   $f_{xx}(x_0,y_0) > 0$ and $f_{yy}(x_0,y_0) > 0$ and $\Delta(x_0,y_0) > 0$

3. For a Saddle point
   $\Delta(x_0,y_0) < 0$

It should be noted that when

$$\Delta(x_0,y_0) = 0,$$

then, this test cannot be used to establish whether the critical point is either a minimum or maximum.

Example

Find the local extrema of the function

$f(x,y)= x^2 + 2xy +2y^2 +2x – 2y$

At critical points

$f_x = 2x + 2y + 2 = 0$……………………………………………………………………..(i)

$f_y = 2x + 4y – 2 = 0$………………………………………………………………………(ii)

Subst in (ii)

$2y = 4 \Rightarrow y = 2$

Subst in (i)

$2x +4 +2 = 0 \Rightarrow x = -3$

The critical point is at (-3,2)

$f_{xx} = 2$, $f_{yy} = 4$ and $f_{xy} = 2$.

$\Delta (x_0,y_0) = f_{xx} f_{yy} - [f_{xy}]^2$

$= 8- 4 =4>0$

$f_{xx} >0$, $f_{yy} >0$ and $\Delta > 0$

Thus the point (-3, 2) is a local minimum and $f(-3,2) = (-3)^2 +2(-3)(2) +2(-3) -2(2) = -5$

Example

The total cost per production run (in thousands of dollars) for Hides and Skins Company limited is given by

$C(x,y) = 3x^2 + 4y^2 – 5xy +3x – 14y + 20$

Where x denotes the number of man-hours and y the number (in thousands) of the pairs of shoes per run. What values of x and y will result in the minimum total cost per production run?

Solution

At the extrema,

$C_x (x,y) = C_y (x,y) =0$

$C_x (x,y) = 6x – 5y + 3 = 0$

$C_y (x,y) = 8y – 5x – 14 = 0$

Thus we have,

6x- 5y = -3x

-5x + 8y = 14

This gives a

x = 2 and y = 3

To establish that this s a minimum we test using the condition,

$f_{xx}(x_0,y_0) > 0$ and $f_{yy}(x_0,y_0) > 0$ and $\Delta(x_0,y_0) > 0$

$f_{xx}(x_0,y_0) = 6>0$, $f_{yy}(x_0,y_0) = 8 > 0$ and

$\Delta(x_0,y_0) = f_{xx}(x,y)\, f_{yy}(x,y) - [f_{xy}(x,y)]^2 = 6(8) - (-5)^2 = 23 > 0$

Therefore 2,000 man-hours and 3,000 pairs will result in minimum cost per production run

Lagrange multipliers

Sometimes maximization and minimization is subject to constraints.This is where Lagrange Multiplier method is used. Suppose we have an interest in extreme values of the function f(x,y,z), subject to a constraint function g(x,y,z)= 0, we can construct an auxillary function

$F(x,y,z,\lambda) = f(x,y,z) - \lambda\, g(x,y,z) = 0$

$F_x = f_x - \lambda g_x = 0$

$F_y = f_y - \lambda g_y = 0$

$F_z = f_z - \lambda g_z = 0$

$F_\lambda = -g_\lambda = 0$

Example

By using L units of labour and K units of capital a Steel Milling limited can produce P bundles of angle bars, where

$P(L, K) = 100L^{1/2} K^{2/3}$

The costs for each bundle is $100 for the unit of labour and $300 for the unit of capital. Steel Milling has a budgetary sum of $ 45,000 available for production of the angle bar bundles. Using the Lagrange Multiplier method, determine the units of labour and capital that the firm should use to maximize production.

Solution

$P(L, K) = 100L^{1/2} K^{2/3}$

The constraining function is

100L + 300 K – 45,000 = 0

The auxiliary function

$P(L,K,\lambda) = 100L^{1/2} K^{2/3} - \lambda\,(100L + 300\,K - 45{,}000)$

To obtain maximum P(L, K) we have

$P_L = \frac{100}{2} L^{-1/2} K^{2/3} - 100\lambda = 0 \Rightarrow \lambda = \frac{1}{2} L^{-1/2} K^{2/3}$ …………………………………………..(i)

$P_K = \frac{200}{3} L^{1/2} K^{-1/3} - 300\lambda = 0 \Rightarrow \lambda = = \frac{2}{9} L^{1/2} K^{-1/3}$……………………………………………(ii)

$P_\lambda = -(100L + 300\,K - 45{,}000) = 0$……………………………………………………….(iii)

From (i) and (ii), we have

$\frac{1}{2} L^{-1/2} K^{2/3} = \frac{2}{9} L^{1/2} K^{-1/3} \Rightarrow K = \frac{4}{9} L$

Subst. in (iii)

$100L + 300(\frac{4}{9} L) - 45{,}000 = 0 \Rightarrow L + \frac{4}{3} L = 450$

=> L = 193 and K = 86.

Approximations

In the case of a single variable function, for sufficiently small Δx, the approximation of

$f(x_0 + \Delta x) = f(x_0) + f^{/}(x_0)\,\Delta x$.

If $z = f(x,y)$, provided Δx and Δy are sufficiently small, then,

$f(x_0 + \Delta x. y_0 + \Delta y) \approx f(x_0, y_0) + f_x(x_0, y_0)\,\Delta x + f_y(x_0, y_0)\,\Delta y$

Example₀

$F(x,y) = \sqrt{x+y} + \sqrt{x-y}$

Determine the estimate of f(10.1, 5.8)

In this case we take $x_0 = 10$ and $y_0 = 6 \Rightarrow \Delta x = 0.1$ and $\Delta y = -0.2$

$f(10, 6) = \sqrt{10+6} + \sqrt{10-6} = 4 + 2 = 6$

$f_x(10,6) = \frac{1}{2\sqrt{10+6}} + \frac{1}{2\sqrt{10-6}} = \frac{1}{8} + \frac{1}{4} = \frac{3}{8}$

$f_y(10,6) = \frac{1}{2\sqrt{10+6}} - \frac{1}{2\sqrt{10-6}} = \frac{1}{8} - \frac{1}{4} = -\frac{1}{8}$

$f(10 + 0.1, 6 - 0.2) \approx 6 + \frac{3}{8}(0.1) + \left(\frac{-1}{8}\right)(-0.2) = 6 + 0.0375 + 0.0250 = 6.0625$

$f(10.1, 5.8) \approx 6.0625$

INTEGRAL CALCULUS

Differential calculus is about rate of change while Integral calculus is about accumulation as a result of change. For example, if s(t) is a distance covered after time t, then the speed at time t is given by $s^{/}(t)$. It should be noted that the two are related. If one is known then the other can be determined.

The process of finding a function when its derivative is known is called **integration** and the function to be found is called the **anti-derivative or the integral** of the given function. So the process of integration is the **reverse** of differentiation.

Example 1

We can say that

$x^2 + c$ is the integral of 2x with respect to x

Note that the constant c cannot be determined unless there is other data known about the behaviour of the integral. It is called the constant of integration.

Example 2

Consider $\frac{dy}{dx}$ = x,

then,

$$y = \frac{1}{2}x^2 + c$$

Example 3

Consider $\frac{dy}{dx}$ = ax + b

then

$$y = \frac{a}{2}x^2 + bx + c$$

Since c is arbitrary, the integral obtained is called an **Indefinite Integral.** In general, if F(x) is an anti-derivative or Integral of f(x), we say that

$$\int f(x)\,dx = F(x) + c$$

The sign $\int \ldots\ldots\ldots dx$ indicates the integral of the any expression …. With respect to x. In the above expression, **f(x)** is referred to as the **integrand** and **dx** is the **differential** of the variable and **F(x)** is the **integral.**

The integral is the inverse of

$$\frac{d(\ldots\ldots)}{dx}$$

which means the derivative of the bracketed expression with respect to x.

Power formula of Integration

$$\int \boldsymbol{x^n dx} = \frac{\boldsymbol{x^{n+1}}}{\boldsymbol{n+1}} + \mathbf{c}, \qquad \forall\ \mathbf{n \neq -1}$$

Note that

$$\frac{d\left(\frac{x^{n+1}}{n+1}\right)}{dx} = \frac{1}{n+1}\frac{d(x^{n+1})}{dx} \qquad \forall\ n \neq -1$$

=> Derivative of $\frac{\boldsymbol{x^{n+1}}}{\boldsymbol{n+1}}$ is x^n.

Example 1

$\int \boldsymbol{x^4 dx} = \frac{\boldsymbol{x^{4+1}}}{\boldsymbol{4+1}} + \mathbf{c} = \frac{\boldsymbol{x^5}}{\boldsymbol{5}} + \mathbf{c},$

Example 2

$\int \frac{1}{x^2}\, dx = \frac{x^{-2+1}}{-2+1} + c = \frac{x^{-1}}{-1} + c = -\frac{1}{x} + c$

Example 3

$\int dx = \frac{x^{0+1}}{0+1} + c = x + c$

Example 4

$\int \frac{1}{x}\, dx = \ln x + c$ (explanation $y = \ln x \Rightarrow x = e^y$. $\frac{d(x)}{dx} = \frac{d(e^y)}{dx} = e^y\frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{1}{e^y} = \frac{1}{x}$)

$$\int \frac{1}{x^2}\, dx = \frac{x^{-2+1}}{-2+1} + c = \frac{x^{-1}}{-1} + c = -\frac{1}{x} + c$$

Standard Elementary Integrals

$$\int x^n dx = \frac{x^{n+1}}{n+1} + c, \qquad \forall\, n \neq -1$$

$$\int \frac{1}{x} dx = \ln x + c \qquad \forall\, x \neq 0$$

$$\int e^x dx = e^x + c.$$

Example

$\int cf(x)dx = c\int f(x)dx$

$\int [f(x) + g(x)]dx = \int f(x)dx + \int g(x)dx$

Illustration 1

$\int \left(x - \frac{2}{x}\right)^2 dx = \int x^2 - 2x\,.\frac{2}{x} + \frac{4}{x^2} dx = \int x^2\, dx - 4\int dx + \int \frac{4}{x^2} dx$

$= \frac{x^3}{3} - 4x - \frac{4}{x} + c$ (note $\int \frac{4}{x^2} dx = 4\,\frac{x^{-2+1}}{-2+1} = 4.\,\frac{x^{-1}}{-1} = -\frac{4}{x}$)

Illustration 2

$\int \left(\frac{2-4x+x^2+2x^3}{x^2}\right) dx = \int \frac{2}{x^2} dx - 4\int \frac{1}{x}\, dx + \int dx + 2\int x\, dx = -\frac{2}{x} - 4\ln|x| + x + x^2 + c$

Examples of Application

1. The marginal Revenue function is
   $R'(x) = 10 - 0.02x$
   a) Find the revenue function
      Solution
      The revenue function

$R(x) = \int (R^{/}(x))dx = \int (10 - 0.02x)dx = 10x - 0.01x^2 + c$
When x = 0 R(x) = 0 => $10(0) - 0.01(0)^2 + c = 0$ => c = 0
Therefore the revenue function
$R(x) = 10x - 0.01x^2$

b) Find the demand relation of the firm's product
Given the product price as p, we can say that the revenue
R(x) = price (p) . quantity supplied(x)
=> R(x) = px
Therefore $px = 10x - 0.01x^2$
Dividing both sides by x , we have,
P = 10 – 0.01x, which is the demand curve,

2. The marginal profit function of a firm is $P^{/}(x) = 5 - 0.002x$ and the firm made a profit of 310 when 100 un8its were sold. What is the profit function of the firm?
$P(x) = \int (P^{/}(x))dx = \int (5 - 0.002x)dx = 5x - 0.001x^2 + c$
When x = 100. P =310,
=> $310 = 5(100) - 0.001(100)^2 + c$ => 310 = 500 – 10 + c => c = 310 – 490 = - 180
Hence,
$P(x) = 5x - 0.001x^2 - 180$

Definite Integral

A **definite integral** of a function f(x) from x=a to x= b is denoted as $\int_a^b f(x)dx$ = F(b) –F(a). From this we note

f(x) is the integrand,

F(x) is the integral,

F(a) and F(b) are the values of the integral at x= a and x= b respectively,

a and b are the lower and upper limits of the integral respectively.

We say that $\int_a^b f(x)dx$ is a definite integral because it has limits and its valuation the integral constant cancels out as shown below:

$$\int_a^b f(x)dx = [F(x)]_a^b = [F(b) + c] - [F(a) + c] = F(b) - F(a).$$

Example 1

Evaluate the following definite integral:

a) $\int_1^3 x^3 dx$
$\int_1^3 x^3 dx = \left[\frac{1}{4}x^4\right]_1^3 = \frac{1}{4}[3^4 - 1^4] = 81 - 1 = 80$

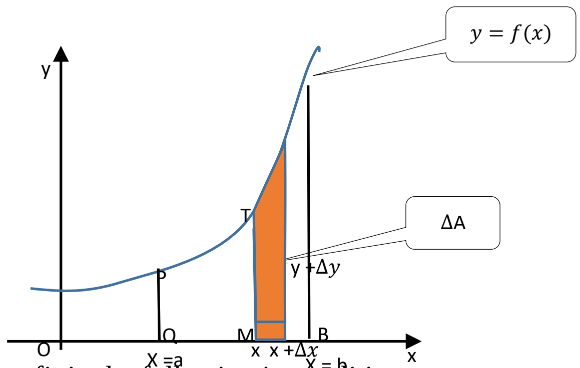
b) $\int_2^5 2x + 4\, dx = \left[2\frac{x^2}{2} + 4x\right]_2^5 = [5^2 + 4(5)]_2^5 - [2^2 + 4(2)]_2^5$

= 4 + 8 = 12

c) $\int_o^2 e^{2x}dx = \left[\frac{1}{2}\ e^{2x}\right]_1^3 = \frac{1}{2}\,[e^{2(3)} - e^{2(0)}] = \frac{1}{2}\,[e^6 – 1] = \frac{1}{2}\,[403.43 – 1] = 201.215.$

Evaluation Area under a curve

Consider the graph



The shaded area ΔA can fit in the following inequalities:

$$y.\,\Delta x < \Delta A(x) < (y + \Delta y).\,\Delta x$$

Dividing all through by Δx

$$y < \frac{\Delta A}{\Delta x} < (y + \Delta y)$$

$(y + \Delta y) \rightarrow y$ as $\Delta x \rightarrow 0$

When we consider the limits of the inequalities as $\Delta x \rightarrow 0$, we have

$\lim_{\Delta x \rightarrow 0} \frac{\Delta A}{\Delta x}$ = y or $A^{/}(x)$ = y => $\frac{dA}{dx}$ = y = f(x). We therefore have,

$$A = \int y\ dx\ = \int f(x)\ dx$$

Let F(x) be the anti-derivative of f(x) then

A = F(x) + c

Now if we move TM to PQ, where x = a, then area under the curve will A = 0 =>F(a) + c =0.

=> c = -F(a).

Therefore A = F(x) – F(a)

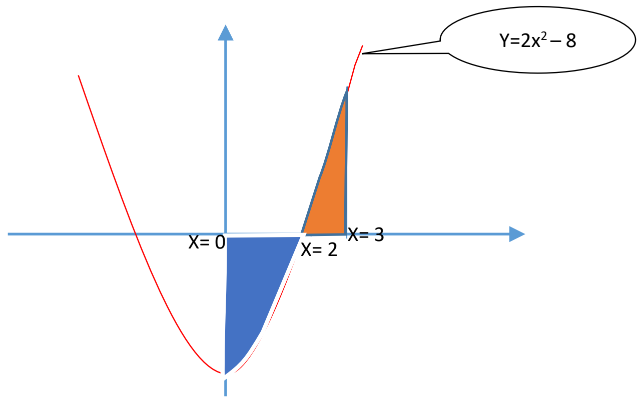
When x = b at B , the area under the curve A = F(b) – F(a) = $\int_a^b f(x)dx$ which is a definite integral. This put differently means

$$\mathbf{A} = \int_a^b y\,dx$$

Example

Find the area bounded by the x-axis, the curve y = f(x) = $2x^2 - 8$ and the lines x =0 and x = 3.

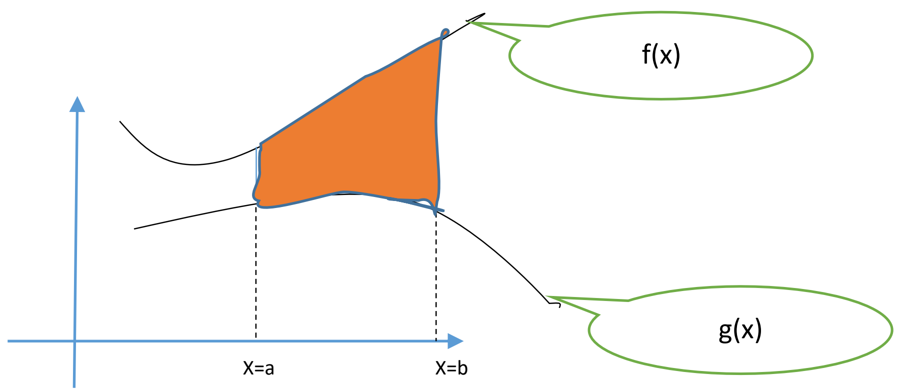
To answer this thee a need to plot the curve.



The area bounded by the x-axis, the curve y = f(x) = $2x^2 - 8$ and the lines x =0 and x = 3 is given by

$$A = -\int_0^2 2x^2 - 8\,dx + \int_2^3 2x^2 - 8\,dx = -\left[\frac{2}{3}x^3 - 8x\right]_0^2 + \left[\frac{2}{3}x^3 - 8x\right]_2^3$$

$$= -\left[\frac{16}{3} - 16\right] + \left[\left[\frac{54}{3} - 24\right] - \left(\frac{16}{3} - 16\right)\right] = 24 - \frac{32}{3} = \frac{40}{3}$$



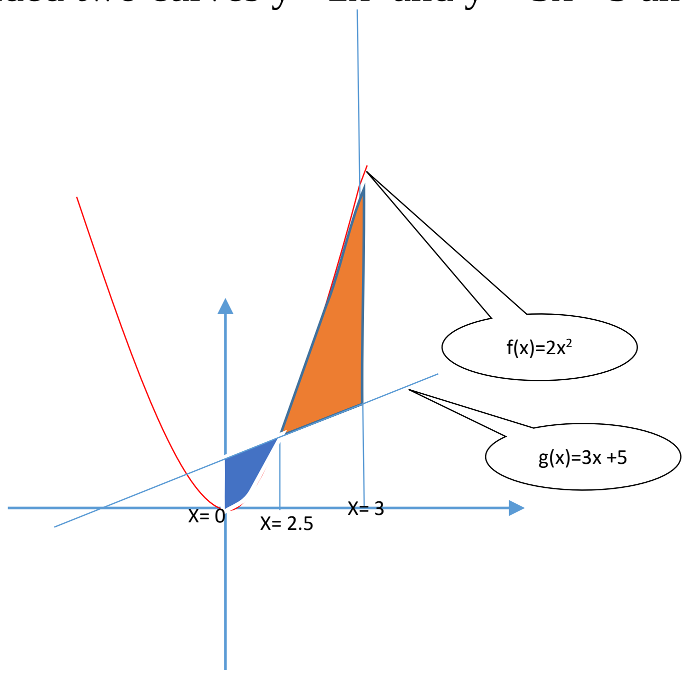
Area between two curves

The required area of the shaded part between the two curves is given by,

$A = \int_a^b f(x) - g(x)dx.$

Example,

Find the area bounded two curves y= $2x^2$ and y = 3x +5 and the lines x=0 and x=3



The area bounded by the, the curves y = f(x) = $2x^2$ and y= 3x + 5 and the lines x =0 and x = 3is given by

$A = \int_a^b f(x) - g(x)dx$. However we need to assess the trend of the relationship of the curves.

We note that the two graphs meet when

$$2x^2 = 3x + 5 \Rightarrow 2x^2 - 3x - 5 = 0$$

$$\Rightarrow 2x^2 + 2x - 5x - 5 = 0 \Rightarrow 2x(x+1) - 5(x+1) = 0$$

Hence the two graphs intersect at (-1, 2) and (2.5, 12.5). In this case the point of interest is (2.5, 12.5).

Therefore

$$A = \left[\int_0^{2.5} (3x+5) - 2x^2 dx\right] + \left[\int_{2.5}^{3} 2x^2 - (3x+5)dx\right]$$

$$= \left[\frac{3}{2}x^2 + 5x - \frac{2}{3}x^3\right]_0^{2.5} + \left[\frac{2}{3}x^3 - \frac{3}{2}x^2 - 5x)\right]_{2.5}^{3}$$

$$= \left[\frac{3}{2}(2.5)^2 + 5(2.5) - \frac{2}{3}(2.5)^3\right] + \left[\frac{2}{3}(3)^3 - \frac{3}{2}(3)^2 - 5(3)\right] - \left[\frac{2}{3}(2.5)^3 - \frac{3}{2}(2.5)^2 - 5(2.5)\right]$$

$$= 3(2.5)^2 - \frac{4}{3}(2.5)^3 + 10(2.5) + 18 - \frac{27}{2} - 15$$

=18.75 – 20.83 + 25 +18 – 13.5 - 15 = 12.42.

Example

Marginal cost of a certain firm is given by $C^{/}(x) = 24 – 0.03x + 0.006x^2$ if the cost of producing 200 units is UGX 22,700 find,

(a) The cost function,
(b) The fixed cost,
(c) The cost of producing 500 units

Example,

Marginal cost of a certain firm is given by $C^{/}(x)$ =15.7 – 0.002x whereas the marginal revenue is

$R^{/}(x)$ =22 - 0.004x. Determine the increase in profit when the sales are increased from 500 to 600 units.

Example

The revenue and cost rates for an oil drilling operation are given by $R^{/}(x) =14 – t^{1/2}$ and

$C^{/}(x) = 2 + 3t^{1/2}$ respectively, where t is time in years and R and C are in millions of dollars. How should the drilling be continued to obtain maximum profit? What is the maximum profit?

Example

After a person has been working for t hours on a particular machine, x units will have been produced, where the rate of production(number of units per hour) is given by

$$\frac{dx}{dt} = 10(1 - e^{-t/50}).$$

How many units are produced during the person's first 50 hours on the machine? How many are produced during the second 50 hours.

Integration by substitution

This method of integration amounts to the chain rule in reverse. It is also the generalization of the power rule in the reverse. We know that

$$\int u^n\,du = \frac{u^{n+1}}{n+1} + c \qquad (n \neq -1)$$

If we put u = g(x), then du = $g^{/}(x)dx$, and we can write that

$$\int [g(x)]^n\, g^{/}(x)dx = \frac{(g(x))^{n+1}}{n+1} + c$$

Example 1

$$\int 2(2x+1)^5\,dx$$

If we put u = $2x + 1$ then du = 2, therefore

$\int 2(2x+1)^5\,dx = \int 2u^n\,du = 2\left(\frac{u^{5+1}}{5+1}\right) = \frac{1}{3}(2x+1)^6 + C$

Example 2

$$\int 2x\sqrt{1+\ x^2}\,dx$$

=

$$\int 2x\,(1+\ x^2)^{1/2}\,dx$$

Let u = 1 + x^2 => du = 2x dx , so can write

$\int 2x\,(1+\ x^2)^{1/2}\,dx = \int (u)^{1/2}\,du = \frac{2}{3}(u)^{3/2} + C = \frac{2}{3}(1+x^2)^{3/2} + C$

Note:

1. Identify the existence of a function and its derivative,
2. The differential dx along with the rest of the integrand are transformed or replaced by u and du,
3. After integration the constant is added,
4. A final substitution is necessary to write in terms of variable x.

Example 3

A) Evaluate the following integral

$$\int \mathbf{4x(2x^2+1)^5\,dx}$$

Using the method of integration by substitution, we have

Let u = $2x^2 + 1$ => du = 4x dx , so can write

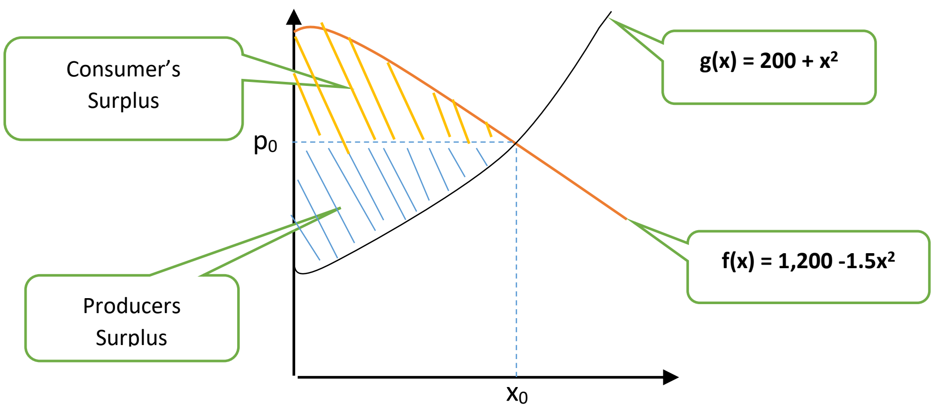
$\int 4x\,(2\,x^2+1)^5\,dx = \int (u)^5\,du = \frac{1}{6}(u)^6 + C = \frac{1}{6}(2x^2+1)^6 + C$

B) A company has established that the supply and demand curves for its product are

Supply: **200 +** $\mathbf{x^2}$
Demand: **1,200 -1.5**$\mathbf{x^2}$

(i) Illustrate graphically, areas showing the consumer's Surplus and Producer's surplus, hence determine their values.

Solution



(ii) In order to establish the Producer's Surplus (P.S) and Consumer's Surplus (C.S.), there is a need to establish the point (x_0, p_0) which is the market equilibrium. That is

$1{,}200 - 1.5x^2 = 200 + x^2 \Rightarrow 2.5x^2 - 1000 = 0 \Rightarrow x = 20$ or -20.
We cannot have negative number of products and therefore the feasible answer is $x_0 = 20$
$\Rightarrow p_0 = 200 + 400 = 600$.

C.S. = $\int_0^{x_0} f(x) - p_0)dx = \int_0^{20}(1{,}200 - 1.5x^2) - 600)dx = \left[600x - \frac{1.5}{3}x^3\right]_0^{20}$
$= 12{,}000 - 4{,}000 = 8{,}000$.
P.S. = $p_0x_0 - \int_0^{x_0} g(x)dx = (20x600) - \int_0^{20}(200 + x^2)dx = 12{,}000 - \left[200x + \frac{1}{3}x^3\right]_0^{20}$
$= 12{,}000 - [4{,}000 + \frac{8{,}000}{3}] = \frac{16{,}000}{3}$.

C) From operation records of Igara Shoe manufacturers Ltd, the marginal cost, in dollars, was found to be,

$$\mathbf{C'(x) = x\sqrt{x^2 + 2500}}$$

where x is the numbers of pairs produced per week. If the fixed costs per week are **$ 100**, find the cost function.

Solution

The cost function C(x) = $\int C'(x)dx = \int x(x^2 + 2500)^{1/2}\,dx$

Using substitution method of integration, let u = $x^2 + 2500$ => du = 2x dx
Therefore

$\int x(x^2 + 2500)^{1/2}\, dx = \int \frac{(U)^{1/2}}{2}) dx = \frac{(U)^{3/2}}{3} + C$

=> $\int x(x^2 + 2500)^{1/2}\, dx = \frac{(x^2+2500)^{3/2}}{3} + C$

The fixed costs are when x = 0, therefore

$\frac{(0^2+2500)^{3/2}}{3} + C = 100 \Rightarrow C = 100 - \frac{(2500)^{3/2}}{3} = 100 - \frac{(50)^3}{3} = \frac{300-125{,}000}{3}$

$= \frac{-124{,}700}{3}$

Hence the cost function, $C(x) = \frac{(x^2+2500)^{3/2}}{3} - \frac{124{,}700}{3}$

D) The daily marginal profit of a firm is given by $P'(x) = -2 + \frac{x}{\sqrt{x^2+900}}$. If the firm loses $130 per day when it sells only 40 units per day, find the firm's profit function.

Integration by parts

This is a consequence of the derivative of a product of functions.

$\frac{d(v(x)U(x))}{dx} = v(x)u'(x) + u(x)v'(x) \Rightarrow v(x)u'(x) = \frac{d(v(x)U(x))}{dx} - u(x)v'(x)$

$\int v(x)u'(x)\, dx = u(x)v(x) - \int u(x)v'(x)dx$

Or

$$\int v \frac{du}{dx} dx = uv - \int u \frac{dv}{dx} dx$$

So if we say,

$v(x) = f(x)$ then $v'(x) = f'(x)$ and if $u'(x) = g(x)$ then $u(x) = G(x)$ where $G(x)$ is the integral of $g(x)$.

We can write,

$\int f(x)g(x)dx = f(x)\, G(x) - \int f'(x)G(x)dx$

Example

$\int x\, e^{2x}\, dx$

Let $f(x) = x$ and $g(x)= e^{2x} \Rightarrow f'(x) = 1$ and $G(x) = \frac{1}{2} e^{2x}$

Therefore,

$\int x\, e^{2x}$ dx = $\frac{x}{2}$ e^{2x} - $\int 1.\frac{1}{2}\ e^{2x}$ dx

= $\frac{x}{2}$ e^{2x} - $\frac{1}{4}$ e^{2x} + C

= $\frac{1}{4}$ (2x – 1) e^{2x} + C

Note that if you chose f(x) = e^{2x} , the new integral to the right would become difficult to integrate.

Example 2

$\int x^2\ (lnx)$ dx

Let g(x) = x^2 => G(x) = $\frac{1}{3}x^3$

f(x) = lnx => $f^{/}$(x) = $\frac{1}{x}$

Applying

$\int f(x)g(x)dx$ = f(x) G(x) - $\int f^{/}(x)G(x)dx$

=> $\int x^2\ (lnx)$ dx = $\frac{1}{3}x^3$ lnx - $\int \frac{1}{x}\frac{x^3}{3}dx$ = $\frac{1}{3}x^3$ lnx - $\frac{x^3}{9}$ + C

Rules of integration by parts

1. If the integrand is a product of a polynomial in x and an exponential function, then it is often useful to take f(x) ie v(x) as the given polynomial,
2. If the integrand contains a logarithmic function as a factor , it is often useful to choose this as f(x) ie v(x) . If the integrand consists entirely a logarithmic function, we can take g(x) ie $\frac{du}{dx}$ as 1.

Example

$\int \ \ln(2x-1)$dx

f(x) = $\ln(2x-1)$ => $f^{/}$(x) = $\frac{2}{2x-1}$, g(x) = 1 => G(x) = x

$\int \ \ln(2x-1)$dx = x $\ln(2x-1)$ - $\int \ \frac{2x}{2x-1}$dx = x $\ln(2x-1)$ - $\int \ 1+\frac{1}{2x-1}$dx

= x $\ln(2x-1)$ – x – $\frac{1}{2}$ln(2x-1) + C = $\frac{1}{2}$(2x-1) $\ln(2x-1)$ – x + C

Exercise

1. $\int x\ ^2 lnx\ dx$
2. $\int x\ (lnx)^2\ dx$
3.
4. $\int x^5\ e^{x^3}\ dx$ Hint write $x^5\ e^{x^3}$ as $(x^3(x^2)\ e^{x^3}$ then let $\frac{du}{dx}$ = $x^2\ e^{x^3}$ use Integration by parts and then by substitution.
5. Evaluate $\int x^2\ (lnx)^2\ dx$

6. A firm has marginal cost per unit of its product given by

$$C^{/}(x) = \frac{1000\ln(2x+40)}{(2x+40)^2}$$

Where x is the level of production. If the fixed costs are \$4000. Determine the cost function.

7. The marginal revenue of a firm is $R^{/}(x) = 10(20 - x)\, e^{-x/20}$. Find the revenue function and the demand equation of the product.